Actuarial Mathematics for Life Contingent Risks

The substantially updated third edition of the popular *Actuarial Mathematics for Life Contingent Risks* is suitable for advanced undergraduate and graduate students of actuarial science, for trainee actuaries preparing for professional actuarial examinations, and for life insurance practitioners who wish to increase or update their technical knowledge. The authors provide intuitive explanations alongside mathematical theory, equipping readers to understand the material in sufficient depth to apply it in real world situations and to adapt their results in a changing insurance environment. Topics include modern actuarial paradigms, such as multiple state models, cash flow projection methods and option theory, all of which are required for managing the increasingly complex range of contemporary long-term insurance products.

Numerous exam-style questions allow readers to prepare for traditional professional actuarial exams, and extensive use of Excel ensures that readers are ready for modern, Excel-based exams and for the actuarial work environment. The Solutions Manual (ISBN 9781108747615), available for separate purchase, provides detailed solutions to the text's exercises.

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ACTUARIAL MATHEMATICS FOR LIFE CONTINGENT RISKS

THIRD EDITION

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To Carolann, Vivien and Phelim

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Preface to the third edition

Life insurance has undergone enormous change in the last two to three decades. New and innovative products have been developed at the same time as we have seen vast increases in computational power. In addition, the field of finance has experienced a revolution in the development of a mathematical theory of options and financial guarantees, first pioneered in the work of Black, Scholes and Merton, and actuaries have come to realize the importance of that work to risk management in actuarial contexts.

In this book we have adapted the traditional approach to the mathematics of life contingent risks to account for the products, science and technology that are relevant to current and future actuaries, taking into consideration both demographic and financial uncertainty. The material is presented with a certain level of mathematical rigour; we want readers to understand the principles involved, rather than to memorize methods or formulae. The reason is that a rigorous approach will prove more useful in the long run than a short-term utilitarian outlook, as theory can be adapted to changing products and technology in ways that techniques, without scientific support, cannot. However, this is a very practical text. The models and techniques presented are versions, a little simplified in parts, of the models and techniques in use by actuaries in the forefront of modern actuarial management.

The first seven chapters set the context for the material, and cover traditional actuarial models and theory of life contingencies, with modern computational techniques integrated throughout, and with an emphasis on the practical context for the survival models and valuation methods presented. Through the focus on realistic contracts and assumptions, we aim to foster a general business awareness in the life insurance context, at the same time as we develop the mathematical tools for risk management in that context.

From Chapter 8, we move into more modern theory and methods.

In Chapter 8 we introduce multiple state models, which generalize the life–death contingency structure of previous chapters. Using multiple state models allows a single framework for a wide range of insurance, including income replacement insurance, where benefits and premiums depend on the health status of the policyholder, and critical illness insurance, which pays a benefit on diagnosis of certain serious medical disorders. We also present other applications of multiple state models, including long-term care, continuing care retirement communities and structured settlements.

In Chapter 9 we consider a particular type of multiple state model, namely the multiple decrement model, which occurs frequently in actuarial applications, a notable example being in pension plan valuation.

In Chapter 10 we apply the models and results from multiple state models to insurance involving two lives, typically domestic partners. It is increasingly common for partners to purchase life insurance cover or annuity income products where the benefits depend on both lives, not on a single insured life.

In Chapter 11 we apply the theory developed in the earlier chapters to problems involving pension benefits. Pension mathematics has some specialized concepts, particularly in funding principles, but in general this chapter is an application of the theory in the preceding chapters.

In Chapter 12 we move to a more sophisticated view of interest rate models and interest rate risk. In this chapter we explore the crucially important difference between diversifiable and non-diversifiable risk.

In Chapter 13 we introduce a general algorithm for projecting the emerging surplus of insurance policies, by considering the year-to-year net cash flows. One of the liberating aspects of the computer revolution for actuaries is that we are no longer required to summarize complex benefits in a single actuarial value; we can go much further in projecting the cash flows to see how and when surplus will emerge. This is much richer information that the actuary can use to assess profitability and to better manage portfolio assets and liabilities. In life insurance contexts, the emerging cash flow projection is often called 'profit-testing'.

In Chapter 14 we follow up on the cash flow projections of Chapter 13 to show how profit testing can be used to analyse Universal Life insurance, which is very popular in North America.

In Chapter 15 we use the emerging cash flow approach to assess equity-linked contracts, where a financial guarantee is commonly part of the contingent benefit. The real risks for such products can only be assessed taking the random variation in potential outcomes into consideration, and we demonstrate this with Monte Carlo simulation of the emerging cash flows.

The products that are explored in Chapter 15 contain financial guarantees embedded in the life contingent benefits. Option theory is the mathematics of valuation and risk management of financial guarantees. In Chapter 16 we introduce the fundamental assumptions and results of option theory.

In Chapter 17 we apply option theory to the embedded options of financial guarantees in insurance products. The theory can be used for pricing and for determining appropriate reserves, as well as for assessing profitability.

In Chapter 18 we move into a different aspect of actuarial work and discuss some of the techniques that are used to estimate the survival models that appear in earlier chapters.

In Chapter 19 we present a very brief introduction to the important practical topic of modelling longevity through stochastic mortality models.

The material in this book is designed for undergraduate and graduate programmes in actuarial science, for those self-studying for professional actuarial exams and for practitioners interested in updating their skill set. The content has been designed primarily to prepare readers for practical actuarial work in life insurance and pension funding and valuation. The text covers all of the most recent syllabus requirements for the LTAM exam of the Society of Actuaries and for the CM1 exam of the UK Institute and Faculty of Actuaries. Some of the topics in this book are not currently covered by those professional exams, and many of the topics that are in the exams are covered in significantly more depth in the text, particularly where we believe the content will be valuable beyond the exams.

Students and other readers should have sufficient background in probability to be able to calculate moments of functions of one or two random variables, and to handle conditional expectations and variances. We assume familiarity with the binomial, uniform, exponential, normal and lognormal distributions. Some of the more important results are reviewed in Appendix A. Readers are also assumed to have a knowledge of maximum likelihood estimation, also reviewed in Appendix A. We also assume that readers have completed an introductory level course in the mathematics of finance, and are aware of the actuarial notation for interest, discount and annuities-certain.

Throughout, we have opted to use examples that liberally call on spreadsheet-style software. Spreadsheets are ubiquitous tools in actuarial practice, and it is natural to use them throughout, allowing us to use more realistic examples, rather than having to simplify for the sake of mathematical tractability. Other software could be used equally effectively, but spreadsheets represent a fairly universal language that is easily accessible. To keep the computation requirements reasonable, we have ensured that all but one of the examples and exercises can be completed in Microsoft Excel, without needing any VBA code or macros. Readers who have sufficient familiarity to

write their own code may find more efficient solutions than those that we have presented, but our principle is that no reader should need to know more than the basic Excel functions and applications. It will be very useful for anyone working through the material of this book to construct their own spreadsheet tables as they work through the first seven chapters, to generate mortality and actuarial functions for a range of mortality models and interest rates. In the worked examples in the text, we have worked with greater accuracy than we record, so there will be some differences from rounding when working with intermediate figures.

One of the advantages of spreadsheets is the ease of implementation of numerical integration algorithms. We assume that students are aware of the principles of numerical integration, and we give some of the most useful algorithms in Appendix B.

The material in this book is appropriate for three one-semester courses. The first six chapters form a fairly traditional basis, and would reasonably constitute a first course. Chapters 7–12 introduce more contemporary material, and could be used for the second course. Chapter 11, on pension mathematics, is not required for subsequent chapters, and could be omitted if a single focus on life insurance is preferred. Chapters 13–17 form a coherent, cash-flow-based coverage of variable insurance, which could be the basis of the third, more advanced course. Chapter 18 can reasonably be covered at any point after Chapter 8, and Chapter 19 at any point after Chapter 5.

Changes from the second edition

The text has been updated to reflect changes in insurance and pension benefits since the first edition was published in 2009. In particular, we illustrate how the methods and models covered can be applied to a wide range of newer insurance contracts contingent on morbidity rather than just mortality. Examples include critical illness and long-term care insurance. We have expanded the pension valuation material to give more detail on career average earnings plans, which have become much more popular in the past decade. We have included a chapter (Chapter 18) on how the models that we use throughout the book are developed from demographic data. Finally, we have included discussion of issues around changing mortality, considering deterministic and stochastic models of mortality improvement.

- The first chapter has been significantly expanded to provide readers with greater background about life insurance practice and products. New material has been included on topics such as health insurance, continuing care retirement communities and structured settlements.

- Chapter 3 includes new material about deterministic modelling of mortality improvement and the construction of mortality improvement scales.
- Chapter 8 contains new material on state-dependent annuity and insurance functions, as well as recursions for state-dependent policy values. We now also show how multiple state models can be applied to topics such as critical illness insurance, long-term care and structured settlements.
- Chapter 9 contains material on multiple decrement models that was in Chapter 8 of the second edition. As a result, Chapters 10–16 of the second edition appear as Chapters 11–17 in this edition.
- Chapter 11 has been expanded to include updated material on the valuation and funding of pension plan benefits and new content on the valuation and funding of retiree health benefits.
- Chapters 13 and 14 have been rearranged. Chapter 14 now covers Universal Life insurance. Some of the material on participating insurance that was previously in this chapter has been moved to Chapter 13.
- Chapter 18 is a new chapter dealing with estimation for lifetime distributions and multiple state transition intensities. Consequently, Appendix A has been expanded to include a review of the key points about maximum likelihood estimation.
- Chapter 19 is a new chapter which provides an introduction to the key ideas about the Lee–Carter and Cairns–Blake–Dowd stochastic mortality models.
- The end-of-chapter exercises have been reorganized as short, long and Excel-based questions. We have also added new exercises to almost all chapters.
- In a number of places, particularly Chapters 6 and 8, we have changed exercises to make them more useful for examination preparation, in particular using tables of insurance functions, rather than assuming that readers can access the required functions using an Excel workbook.

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