PART I

GENERAL PROPERTIES OF FIELDS; SCALARS AND GAUGE FIELDS

1 Short Review of Classical Mechanics

In this first chapter, I will quickly review concepts of classical mechanics that we will need in order to understand classical field theory. Classical field theory is a generalization of classical mechanics, so by understanding well enough the concepts of classical mechanics, we will be ready for its generalization. We will also need to understand a bit about quantum mechanics, though we will deal (mostly) with classical concepts in this book, leaving the quantum concepts for quantum field theory.

1.1 A Note on Conventions

In most of this book, I will use field theorist's conventions, with $\hbar = c = 1$, unless needed to emphasize some quantum or (non)relativistic issues. We can always reintroduce $\hbar$ and c by dimensional analysis, if needed. In these conventions, there is only one dimensionful unit, namely $mass = 1/length = energy = 1/time = \dots$. When I speak of dimension of a quantity, I refer to mass dimension.

For the Minkowski metric $\eta^{\mu\nu}$, I use the mostly plus signature convention, so in the most relevant case of 3+1 dimensions, the signature is $(-+++)$, for $\eta^{\mu\nu} = diag(-1,+1,+1,+1)$.

I also use the Einstein summation convention – i.e., repeated indices are summed over. The repeated indices will be one up and one down, unless we are in Euclidean space, when it doesn't matter, so we can put all indices down.

1.2 Lagrangean and Equations of Motion

In classical mechanics, one deals with a collection of particles i, $i = 1,\dots,N$, with positions $\vec{r}_i$ and velocities $\vec{v}_i = \frac{d\vec{r}_i}{dt}$. But in order to construct a useful classical mechanics description, we need to describe the system more abstractly, by constructing independent variables (generalized coordinates) $q_1,\dots,q_k,\dots,q_n$. Then we have

$$\vec{r}_i = \vec{r}_i(q_1,\dots,q_n,t); \quad i = 1,\dots,N, \tag{1.1}$$

and, of course, $3N \geq n$, since we have at most as many variables as positions in three-dimensional space. Then the velocities are, by the chain rule,

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial \vec{r}_i}{\partial t}\,, \tag{1.2}$$

and, therefore, the kinetic energy of the collection of particles is

$$T = \sum_{i=1}^{N} \frac{m_i \vec{v}_i^2}{2} = T(q_1, \ldots, q_n, \dot{q}_i, \ldots, \dot{q}_n; t). \tag{1.3}$$

We consider the case when there are forces, and they are conservative, i.e., they can be obtained from a potential V,

$$\vec{F}_i = -\vec{\nabla}_i V. \tag{1.4}$$

The potential is a function of the coordinates, thus of the generalized coordinates (and perhaps also explicitly of time)

$$V = V(\vec{r}_1, \ldots, \vec{r}_N; t) = V(q_1, \ldots, q_n; t). \tag{1.5}$$

We could also define generalized forces

$$f_j = -\frac{\partial V}{\partial q_j}. \tag{1.6}$$

To describe the classical mechanics system in a unified way, we must construct the fundamental quantity that describes the system, the *Lagrangean*,

$$L = T - V = L(\dot{q}_1, \ldots, \dot{q}_n; q_1, \ldots, q_n; t). \tag{1.7}$$

From this function, we can calculate the *Euler–Lagrange equations of motion* describing the evolution of the system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0. \tag{1.8}$$

For each abstract coordinate (i.e., degree of freedom) q_k, we have an equation, so that the evolution of the system is completely determined.

Once we construct a Lagrangean, we have a complete description of a classical system, and we only need to find solutions to its equations of motion. Thus, classical mechanics is about the construction of Lagrangeans and finding solutions for them.

Classical field theory will be just a generalization of this formalism to a continuum of degrees of freedom, i.e., instead of having a discrete set of coordinates $\{q_k\}_{k=1,\ldots,n}$, we will have a continuum of them.

But there is an even more important concept, the one of *action*, and the associated Hamilton's principle of least action: we minimize the action, which is a functional of the path $\{q_k(t)\}$, with respect to the path – i.e., we look for the stationary paths, given fixed initial and final boundary conditions. The action is the integral of the Lagrangean,

$$S = \int_{t_1}^{t_2} L dt. \tag{1.9}$$

The action is the fundamental starting point of the classical mechanics and classical field theory analysis and is more important than the Lagrangean, since the latter can be changed by a total derivative dF/dt without changing the equations of motion. The extra term in the

action when doing this change is $\int_{t_1}^{t_2} dt\, dF/dt = F(t_2) - F(t_1)$, which is fixed, so doesn't contribute to the variation of the action.

To derive the equations of motion, we vary S through a variation $\delta q_k(t)$ of the path, given fixed endpoints $q_k(t_1)$ and $q_k(t_2)$, and put the result to zero. Then we obtain

$$\delta S = \sum_{k=1}^{n} \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k + \frac{\partial L}{\partial q_k} \delta q_k \right) = 0. \tag{1.10}$$

We now do a partial integration, using the fact that δ and d/dt commute, so $\delta \dot{q}_k = d(\delta q_k)/dt$, and we obtain

$$\delta S = \sum_{k=1}^{n} \int_{t_1}^{t_2} dt \left[-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial L}{\partial q_k} \right] \delta q_k + \sum_{k=1}^{n} \int_{t_1}^{t_2} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right) , \tag{1.11}$$

and the last term vanishes, as we said, since by our assumption $\delta q_k(t_1) = \delta q_k(t_2) = 0$.

Since the result is supposed to be zero for any variation $\delta q_k(t)$, this can only be true if the integrand multiplying it is zero at all times, i.e.,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0. \tag{1.12}$$

We have thus obtained the Euler–Lagrange equations from the principle of least action.

1.3 Systems with Constraints

Until now, we have assumed that the q_k were independent variables, so the variations with respect to them were considered independent as well. But this need not be the case. Perhaps we cannot solve for independent variables, but we are left with some constraints on the system,

$$\tilde{f}_a(\vec{r}_1, \ldots, \vec{r}_N; t) = 0, \quad \forall a = 1, \ldots, m\,, \tag{1.13}$$

or, in terms of our abstract variables,

$$f_a(q_1, \ldots, q_n) = 0, \quad \forall a = 1, \ldots, m. \tag{1.14}$$

The way to deal with them in the Lagrangean formalism is to introduce *Lagrange multipliers*, which are independent variables that multiply each of the constraints. Indeed, we can add the constraints times the Lagrange multipliers for free to the Lagrangean, since they vanish anyway, thus obtaining the modified action

$$S' = \int_{t_1}^{t_2} dt\, L' = \int_{t_1}^{t_2} dt \left(L + \sum_{a=1}^{m} \lambda_a f_a \right). \tag{1.15}$$

Since the λ_a are independent variables (independent on the $q_k(t)$), we just have an extra term in the Euler–Lagrange equations. Indeed, now we obtain the variation of the action as

$$\delta S = \sum_{k=1}^{n} \int_{t_1}^{t_2} dt \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial q_k} + \sum_{a=1}^{m} \lambda_a \frac{\partial f}{\partial q_k} \right] \delta q_k = 0 \,, \tag{1.16}$$

from which we get the modified Euler–Lagrange equations,

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial q_k}\right) + \frac{\partial L}{\partial q_k} + \sum_{a=1}^{m} \lambda_a \frac{\partial f}{\partial q_a} = 0. \tag{1.17}$$

If before we had n equations (Euler–Lagrange) for n variables $q_k(t)$, now we have $n+m$ equations, n Euler–Lagrange and m constraints $f_a = 0$, for $n+m$ variables, n $q_k(t)$ variables and m Lagrange multipliers λ_a. So by solving them, we fix completely $q_k(t)$ and λ_a.

1.4 Canonically Conjugate Momentum and Conservation Laws

For our abstract variables, we can define a notion of momentum, called the canonical momentum conjugate to q_k,

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}. \tag{1.18}$$

If one of the variables is an actual particle position, $q_k = r_k$, then the Lagrangean is

$$L = m_k \frac{\dot{r}_k^2}{2} + \dots \,, \tag{1.19}$$

so the momentum conjugate to q_k is

$$p_k = \frac{\partial L}{\partial \dot{r}_k} = m\dot{r}_k \,, \tag{1.20}$$

which is indeed the usual particle momentum. That means that in general p_k is a generalization of the momentum, as we expected.

If the Lagrangean is independent on a variable q_k (though of course it can depend $\dot{q}_k$), we have $\partial L/\partial q_k = 0$, and then from the Euler–Lagrange equations we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0 \Rightarrow \frac{d}{dt} p_k = 0 \,, \tag{1.21}$$

i.e., the canonically conjugate momentum is conserved (constant in time). In this case, we say that the variable q_k is *cyclic*.

Reversely, if we have a conserved quantity, we can look for a cyclic independent variable that has it as its canonically conjugate momentum.

The notion of symmetries – and, in particular, symmetries of the Lagrangean (or rather, of the action) – are really fundamental for modern theoretical physics. From the (sometimes abstract) symmetries of the action, we derive many interesting physical properties, so understanding the symmetries of the problem is very important.

We will describe symmetries more in the next chapter, but for the moment we will give a first taste through the simplest and most common case of rotational symmetry. A rotation is a transformation acting linearly on the Cartesian coordinates r_i^a, $a = 1, 2, 3$ (where $\vec{r}_i$ splits as (x_i, y_i, z_i), together r_i^a), as

$$r'^a_i = \Lambda^a{}_b r^b_i \,, \tag{1.22}$$

where $\Lambda^a{}_b$ is a matrix satisfying

$$\Lambda \cdot \Lambda^T = \Lambda^T \cdot \Lambda = \mathbb{1}. \tag{1.23}$$

Such a matrix is called *orthogonal* matrix. Note that here we have used *Einstein's summation convention*, where repeated indices are summed over.

Then we have

$$\vec{r}'^2_i = \sum_a (r'^a_i)^2 = \sum_{a,b,c} (\Lambda^a{}_b r^b_i)(\Lambda_{ac} r^c_i) = \sum_{b,c} r^b_i (\Lambda^T\Lambda)_{bc} r^c_i = \sum_{b,c} r^b_i \delta_{bc} r^c_i = \vec{r}^2_i \,, \tag{1.24}$$

so, indeed, $\vec{r}_i^2$ is invariant. In the same way, we show that

$$\left(\frac{d\vec{r}'_i}{dt}\right)^2 = \left(\frac{d\vec{r}_i}{dt}\right)^2 . \tag{1.25}$$

That means that *if* $V = V(\vec{r}_i^2)$ *only*, i.e., if $V = V(|\vec{r}|)$ and is independent on the particle, a case known as a central potential (for instance, an electron around a nucleus, or a planet around the Sun), then the Lagrangean is rotationally invariant,

$$L(\vec{r}'_i, \dot{\vec{r}}'_i; t) = \sum_i \frac{m_i \vec{v}'^2_i}{2} - V(\vec{r}'^2_i) = \sum_i \frac{m_i \vec{v}^2_i}{2} - V(\vec{r}^2_i) = L(\vec{r}_i, \dot{\vec{r}}_i, t). \tag{1.26}$$

1.5 The Hamiltonian Formalism

The Lagrangean is the difference of the kinetic and potential energies, $L = T - V$, but we know that the total energy is the sum, $E = T + V$. We will, in fact, see that the energy is related to another important concept in classical mechanics, the Hamiltonian. To define it, we need first to understand the concept of Legendre transformation.

Generally, a function $f(x, y)$ has a differential

$$df = u\, dx + v\, dy \,, \tag{1.27}$$

where

$$u \equiv \frac{\partial f}{\partial x}; \quad v \equiv \frac{\partial f}{\partial y}. \tag{1.28}$$

But now we can forget where u, v come from, and consider them as independent variables. In that case, the Legendre transform of the function f is

$$g = f - ux. \tag{1.29}$$

Then its differential is

$$dg = df - u\,dx - x\,du = -x\,du + v\,dy. \tag{1.30}$$

From this differential, we can derive

$$x = -\frac{\partial g}{\partial u}; \quad v = \frac{\partial g}{\partial y}\,, \tag{1.31}$$

which means that the function g is a function of u and y instead,

$$g = g(u,y). \tag{1.32}$$

The Legendre transform was encountered, for instance, in thermodynamics, where the total energy U is a function of extensive quantities, $U = U(S,V,\ldots)$,

$$dU = TdS - PdV + \ldots\,, \tag{1.33}$$

but its Legendre transform in S, the free energy,

$$F = U - TS\,, \tag{1.34}$$

has the differential

$$dF = -SdT - PdV + \ldots\,, \tag{1.35}$$

so is a function of extensive quantities and one intensive one, T,

$$F = F(T,V,\ldots). \tag{1.36}$$

Similarly to the thermodynamics case, we now want to do a Legendre transform of the Lagrangean $L = L(q_k, \dot{q}_k, t)$, with differential

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{1.37}$$

It is a bit unusual that the function depends both on coordinates and their derivatives, so it would be good to do a Legendre transform on the derivatives, to exchange them for independent variables. First, we note that in terms of the conjugate momenta $q_i \equiv \partial L/\partial \dot{q}_i$, the Lagrange equations of motion are

$$\dot{p}_i = \frac{\partial L}{\partial q_i}\,, \tag{1.38}$$

so using them, the Lagrangean differential can be rewritten as

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{1.39}$$

We thus define the Hamiltonian as (minus) the Legendre transform of the Lagrangean,

$$H = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t)\,, \tag{1.40}$$

so its differential is

$$dH = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt. \tag{1.41}$$

The Hamiltonian is then a function of coordinates and their conjugate momenta, plus maybe an explicit time dependence,

$$H = H(q_i, p_i, t). \tag{1.42}$$

From the Hamiltonian differential, we derive the *Hamiltonian equations of motion*,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \tag{1.43}$$

The advantage of the Hamiltonian equations of motion is that the equations are first order in time derivatives, whereas in the Lagrangean case, the equations of motion were quadratic in time derivatives. The disadvantage is that we have doubled the number of variables, instead of q_i, we have now q_i and p_i, which together are called the *phase space*.

If the potential is conservative, independent on path or explicitly on time, $V = V(\{q_i\})$, and so the Lagrangean is independent explicitly on time, and if, moreover, there are no cubic or higher-order terms in the velocity in L, $\dot{q}^n$, $n \geq 3$, then $L = T - V$ implies

$$H = T + V = E\,, \tag{1.44}$$

so the Hamiltonian is the energy of the system.

Hamilton's principle of least action is also modified now, but only by replacing L with its form in terms of H – i.e.,

$$\delta S = \delta \int_{t_1}^{t_2} dt(p_i \dot{q}_i - H(q_i, p_i, t)) = 0. \tag{1.45}$$

1.6 Canonical Transformations

On the phase space, we can make coordinate transformations. If they preserve the Hamiltonian structure, they are called *canonical transformations*. That is, transformations

$$Q_i = Q_i(q_k, p_k, t); \quad P_i = P_i(q_k, p_k, t)\,, \tag{1.46}$$

such that for the new phase space there is a Hamiltonian $\tilde{H}$, with Hamiltonian equations

$$\dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}; \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i}\,, \tag{1.47}$$

and such that we have at the same time the principle of least action for both the original Hamiltonian,

$$\delta \int_{t_1}^{t_2} dt \left(\sum_k p_k \dot{q}_k - H(q_k, p_k, t) \right) = 0\,, \tag{1.48}$$

and the new one,

$$\delta \int_{t_1}^{t_2} dt \left(\sum_i P_i \dot{Q}_i - \tilde{H}(Q_i, P_i, t) \right) = 0\,, \tag{1.49}$$

for any boundary conditions (at t_1, t_2). That means that the integrands must be proportional up to a total derivative term (which doesn't change the equations of motion derived from the action), so we must have

$$\sum_k p_k \dot{q}_k - H(q_k, p_k, t) = \sum_i P_i \dot{Q}_i - \tilde{H} + \frac{dF}{dt}. \tag{1.50}$$

By comparing the coordinate dependence on both sides, we see that the function F is

$$F = F(q_k, Q_i, t). \tag{1.51}$$

We can invert the canonical transformation (it is assumed to be invertible, since we have the same number of coordinates on both sides), and find

$$q_k = q_k(Q_i, P_i); \quad p_k = p_k(Q_i, P_i). \tag{1.52}$$

With a bit more analysis, from the same equality (1.50), we can also find the consistency conditions

$$\begin{aligned} \frac{\partial Q_i}{\partial q_k} &= \frac{\partial p_k}{\partial P_i}\frac{\partial Q_i}{\partial p_k} = -\frac{\partial q_k}{\partial P_i} \\ \frac{\partial P_i}{\partial q_k} &= -\frac{\partial p_k}{\partial Q_i}\frac{\partial P_i}{\partial p_k} = \frac{\partial q_k}{\partial Q_i}. \end{aligned} \tag{1.53}$$

1.7 Poisson Brackets

The concept of Poisson bracket is a very useful one in terms of the theory of the Hamiltonian equations, and the quantization of theories. For a couple of functions of phase space, $f(q_k, p_k)$ and $g(q_k, p_k)$, the Poisson bracket is defined as

$$\{f, g\}_{P.B.} = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} \right). \tag{1.54}$$

As we can see, the Poisson bracket is antisymmetric,

$$\{f, g\}_{P.B.} = -\{g, f\}_{P.B.}, \tag{1.55}$$

just like the commutator.

Since, in particular, the q_k and p_k are functions of phase space, we can define their Poisson brackets, the *fundamental Poisson brackets* on the phase space. We easily find that

$$\begin{aligned} \{q_j, p_k\}_{P.B.} &= \delta_{jk} \{p_k, q_j\}_{P.B.} = -\delta_{jk} \\ \{q_j, q_k\}_{P.B.} &= 0 \{p_j, p_k\}_{P.B.} = 0. \end{aligned} \tag{1.56}$$

Moreover, noticing that the objects appearing in the Hamiltonian equations of motion are of the same type as the ones appearing in the Poisson bracket, we find that the Hamiltonian equations of motion become

$$\dot{q}_i = \{q_i, H\}_{P.B.}; \quad \dot{p}_i = \{p_i, H\}_{P.B.}. \tag{1.57}$$