Invitation to Linear Programming and Game Theory

Written in a conversational tone, this classroom-tested text introduces the fundamentals of linear programming and game theory, showing readers how to apply serious mathematics to practical real-life questions by modeling linear optimization problems and strategic games. The treatment of linear programming includes two distinct graphical methods. The game theory chapters include a novel proof of the minimax theorem for 2×2 zero-sum games. In addition to zero-sum games, the text presents variable-sum games, ordinal games, and n-player games as the natural result of relaxing or modifying the assumptions of zero-sum games. All concepts and techniques are derived from motivating examples, building in complexity, which encourages students to think creatively and leads them to understand how the mathematics is applied. With no prerequisite besides high school algebra, the text will be useful to motivated high school students and undergraduates studying business, economics, mathematics, and the social sciences.

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Preface

This is a text about linear programming and strategic games, mathematical models which are each a type of optimization problem. Linear programming may be viewed as a technique to determine the best way to allocate scarce resources. Game theory is concerned with the best course of action in a situation of strategic conflict. The similarity in these descriptions goes beyond merely that they both give an optimal solution to some problem – it turns out that linear programming can be used to solve certain games. Thus, the two topics are complementary, and it is natural to cover them together. Linear programming and game theory provide two extremely useful lenses through which to regard the world.

The treatment of this material is aimed at first- or second-year undergraduate students, as well as motivated high school students. With these audiences in mind as I wrote, I constrained myself to keep the formal mathematics at a level that would be inviting but not overwhelming. So the tone of the writing is more conversational than some textbooks, and some theorems and propositions are stated without proof, if including the proofs would have been too technical or detract too much from the flow of ideas. Furthermore, the field of mathematical modeling is quite broad, and so another constraint that guided the writing was to make a coherent selection of topics rather than a brief sampling of many different topics.

In the end, a solution that seemed to satisfy these constraints (and one that is hopefully close to optimal) was to focus on deterministic models rather than stochastic ones and to avoid models that required calculus and/or differential equations. Instead, the topics chosen are two examples of *linear models* – those that use the solution of systems of linear equations and matrix algebra. So one perspective for viewing the text is that it presents linear algebra and matrix algebra in action. However, we do not assume that the reader has taken a formal course in linear algebra. Part of the purpose of the text is to introduce topics that might make the further study of linear algebra appealing to the reader.

We can identify at least three distinct groups within the target audience for this text. For the first- or second-year undergraduate student (probably the largest group), a course based around this text would be an excellent *introductory mathematics course for non-science majors*. The applications covered in the text cover a wide range of disciplines such as management and business, economics, political science, and international relations. The mathematics that the students will learn from this text goes beyond minimal quantitative reasoning skills. As such, a course based on this text would probably be better preparation for a future major in management and business, economics, etc., than many "mathematics for liberal arts" courses.

I do not mean to imply that the text would not be useful or appreciated by science majors, but most of these students begin their college-level training in mathematics with the standard calculus track or even with linear algebra. Yet, this text could also be of interest to the potential mathematics major. If the reader enjoys the topics covered here, perhaps the text could serve as a broader invitation than just one to linear programming and game theory; perhaps it could serve as an *invitation to the mathematics major* and provide an entrance into the study of mathematics that is an alternative to the previously mentioned standard calculus track taken by most science majors and mathematics majors.

A second group of students for whom this text was written is the mathematics majors who have already taken a course in linear algebra. For such students, reading this text (either independently or as part of a course) may help fill in a few holes in their training or answer some lingering questions they may have about linear algebra by providing them the opportunity to see linear algebra used in the context of other disciplines.

To cite a specific example of this, linear algebra students learn that vector spaces have many different bases, and they learn how to convert from one basis to another, but it may be lost on those students *why* one would want to convert to a different basis. One answer, perhaps, is given by the study of eigenvectors and diagonalization, for those students who learn those topics. Another answer, which is somewhat less technical than the study of diagonalization, occurs in Chapter 3 of this text, when the reader encounters the solution to linear programming problems via the technique of *graphing in the constraint space*. Each "basic solution" of the problem is obtained precisely by choosing a suitable basis for the constraint space. Readers will find this idea reinforced in Chapter 5, when they realize that every time a pivot is performed in the simplex algorithm, what is really happening is a change from one basis to another. While this text does not cover enough theory to be appropriate as a main text in such a course, it is suitable for consideration as a *supplementary text for linear algebra courses*.

Finally, a third target audience is motivated high school students who are anxious to enrich their education by learning some mathematics that does not appear in the usual K–12 curriculum. The technical requirements for reading this book are minimal, and I believe the style is friendly to the reader who wants to study independently. All that is really assumed in terms of prerequisites is basic high school algebra: how to plot lines and curves and some experience in solving linear systems. It would be helpful in Chapters 8 and Appendix for the reader to also know some very basic set theory – mainly the language of sets and how to form unions, intersections, and complements of subsets. Overall, the prerequisites are very modest.

The material in this text has been classroom tested – I have used the material on matrix algebra and game theory for more than 10 years as the topic for a first-year seminar course in game theory and voting theory at Skidmore College. Prior to that, for several years, I used the linear programming material for an elementary modeling course in mathematics at Skidmore College (part of a series of courses entitled Mathematics in Context), which was designed for liberal arts students with an interest in business, economics, or social sciences. Additionally, I have used nearly all of the material for 18 years as the basis for a course in probability and game theory for the Johns Hopkins University Center for Talented Youth (CTY) program, an intensive summer program for gifted high school students aged 12–16. The material has been successful with both high school and college audiences.

While the focus of the text is on models that are deterministic and linear, nothing in mathematics is truly self-contained. There are so many interconnections between topics that I could not completely exclude stochastic and nonlinear concepts. Stochastic ideas appear in game theory, when probability is used to solve mixed strategy games in Chapters 6–8. However, we keep the discussion of probability to a minimum. In a few pages in Chapter 6, we outline all the probability theory needed for the rest of the book. This approach seemed better than interrupting the flow of ideas for a formal treatment of probability theory. In practice, I have found that most students in my classes at both levels – the college student and the motivated high school student – have either had some exposure to probability in high school or else learn the needed material very quickly; they often have a good intuitive understanding of the topic. Nevertheless, I have placed a more systematic treatment of probability in the Appendix – A Rapid Review of Sets and Probability – for those readers seeking a more complete treatment.

Furthermore, some nonlinear ideas make two brief appearances. The first is in Chapter 1, where there are some exercises in quadratic curves (putting them into standard form) and also some exercises in quadric surfaces (particularly "saddle surfaces" – again putting them into a standard form). These appearances are meant to lend support to the other, linear models that are the main focus of the text. The quadratic curves arise in simple optimization problems that can be compared and contrasted with the later optimization problems that arise in linear programming in Chapter 3. The quadric surfaces are used in Section 6.4, where a novel approach to the minimax theorem is presented for 2×2 constant-sum games. (This novel proof does not extend to games of larger sizes than 2×2, and for the general case, we must rely on linear programming.)

Finally, Sections 8.4 and 8.5 also contain material that lies outside of the linear model focus. The mathematics needed for analyzing n-player games (where $n > 2$) is different from the mathematics of two-player game theory, where matrix algebra is the staple. When $n > 2$, instead of matrix algebra and linear programming, the mathematics used is closer to set theory than it is to linear algebra, another topic that many students have seen or can learn quickly. Even though the topics in this chapter diverge from the linear algebra focus of the rest of the text, including them makes for a somewhat more complete treatment of game theory. Although we only scratch the surface of the theory of n-player games, I felt that I would be remiss to try to introduce the reader to game theory and to pass over n-player games in complete silence.

In short, even with these brief transgressions away from linear models, the focus of our text is narrower than books that attempt to survey many different types of mathematics, such as a typical text in math modeling or finite mathematics. Unlike many of these texts, we do not include such standard topics as financial mathematics, statistics, dynamical systems, graph theory and networks, or logic.

What is gained from omitting these topics is the room to go into greater depth. The treatment of linear programming, for example, includes two different graphical approaches. One is the standard method of graphing in the decision space (often referred to as simply "the graphical method" in most texts), which can handle problems with just two decision variables but an arbitrary number of constraints (problems of size $m \times 2$). But, as mentioned before, we also present a method of graphing in the "constraint space," which can handle problems with just two constraints but an arbitrary number of decision variables (problems of size $2 \times n$.) This is a unique feature of this text. I have never encountered a text at this level that covered this method in detail; indeed, I have only seen it mentioned in one other text (see Loomba, 1976), where it was only briefly covered, and for 2×2 problems only.

Similarly, our treatment of game theory is more complete than what is found in finite mathematics texts, which typically restrict their attention to two-player zero-sum games. We discuss variable-sum games, ordinal games, and also n-player games (Chapter 8); and also go into more detail for zero-sum games than the typical finite mathematics text does, by providing a proof of the minimax theorem.

On the other hand, the treatment of linear programming or game theory here could not compete with more advanced texts that only focus on just one of these topics and/or are targeted at the mathematics major. Linear Programming texts such as Karloff (1991) or Nash and Sofer (1996) are bound to be more complete than the treatment here. Indeed, these texts cover topics such as numerical stability of the simplex algorithm, complexity theory and more sophisticated methods for solving these problems such as the "ellipsoid" algorithm and Karmarkar's algorithm, which we do not even mention. Likewise, game theory texts such as Luce and Raiffa (1957) or the first great treatise on game theory (von Neumann and Morgenstern, 1944), as well as more modern texts such as Taylor and Zwicker (1999), are more comprehensive treatments of game theory than what appears in this text and are aimed at a more mathematically sophisticated reader. Also, there are texts at a higher level that cover both topics, such as Brickman (1989).

This text does not try to compete with these more specialized texts, which address a more mathematically mature reader. For a book at this level, I am comfortable with sacrificing the completeness of those texts in order to keep the readability at a level commensurate with first-year college students, plus or minus a year or two. Additional resources are available online at cambridge.org/9781108476256.

In terms of other texts that are more akin to the present text in style and level, the comparison is closer with the texts by Taylor and Pacelli (2008) or Straffin (1993) for game theory (neither of which, however, covers matrix algebra or linear programming and their applications to games); and to Calvert and Voxman (1989) for linear programming (which omits other linear models such as game theory). Indeed, these three texts were used in early versions of my courses and influenced my understanding of linear programming and game theory to a great extent.

In addition to Calvert and Voxman (1989), I also mention the text of Loomba (1976), which influenced my early thinking on linear programming more than 30 years ago. I am indebted to Alan Taylor and William Zwicker for a number of conversations about game theory and voting theory, for several inspiring lectures I have had the pleasure to hear by each of them, and for recommendations for further reading, especially for pointing me to the work of political scientist Steven Brams in the publications Brams (1994, 1985a, and 1985b). In addition to mathematical influences, music of various genres has always been important to me, and like-minded readers may enjoy the fact that this book is riddled with references in homage to songs and artists.

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