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215 Slenderness I

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Slenderness

Volume 1: Abelian Categories

RADOSLAV DIMITRIC

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This work is dedicated to the
memory of my parents

Nadezhda Blagojevic and Milan Dimitric
[Надежда Благојевић и Милан Димитрић]

Most substantial and worthwhile things
look impossible ... that is,
until they are made or accomplished.

Contents

Preface

Each [book] is a mummified soul embalmed in cere-cloth and natron of leather and printer's ink. Each cover of a true book enfolds the concentrated essence of a man. The personalities of the writers have folded into the thinnest shadows, as their bodies into impalpable dust, yet here are their very spirits at your command.
(Sir Arthur Conan Doyle: *Through the Magic Door*)

The notion of slenderness evolved from intrinsic and interesting observations regarding homomorphisms from infinite products of the integers into the group of integers. Today, slenderness is both a theory and a program, and the present volume is dedicated to demonstrating the ideas and relevant results of the theory and outlining the main aims of the program.

The germ of a well-rounded work begins with a quest of an inquisitive researcher to understand better some mathematical phenomena that interest him. That understanding does not come along a straightforward path, rather through many meanderings through different depths, different heights, and different widths. In this exploration, it is an entirely pleasant experience to get immersed in the voyage and ever-expanding new vistas. Yet, sharing this exploration is also important. Indeed I had this sharing in mind at the very conception of this work, beginning in the mid 1980s. In the end, it turned out that it is harder to present the results to fellow mathematicians than it is to indulge in these research voyages. But, mathematics is a social endeavor and presenting our own work to others is simply a premise of a mathematician's life.

Slenderness is a theory because it now encompasses general results from seemingly disparate areas of algebra, topology, set theory, geometry; and the list is not yet complete. One example of that is in the fact that we can contemplate questions on completeness and completions of objects, through considerations of properties of slenderness, thus bringing in a seemingly fresh approach to completeness, which has been sought after for a long time. Equally significant is the application of slenderness to the questions of large cardinals in set theory. Slenderness is a program that has as one of its goals a classification and characterization of slender objects in general, and in specific categories in particular.

This monograph arose to a large extent from my lecture notes prepared for seminars for advanced graduate students and postdocs that I ran in the period from 1995 to 2001 at the University of California at Berkeley.

Ramifications of the theory of slenderness are rather wide and it was impossible to include all of them in this volume. One of the guiding principles in writing this treatise was not to include too much information and too many facts that would impede analysis and insight into the fundamental results of the theory. On the other hand, I plan to include, in Volume II, a number of topics left out of Volume I (such as discussions on submodules of the infinite product of modules). The next volume will consist primarily of material pertaining to generalizations and dualizations of the theory of slenderness, which in turn open new vistas.

The wonderful reach of slenderness into several mathematical areas is one of the reasons that topological constructs have to be established (in Chapter 1) along with some fundamental results related to inverse limits (Chapter 2). With that groundwork, the general theory of slenderness is introduced in Chapter 3, with further exploration through a still mysterious object $\prod / \coprod$ in Chapter 4. Chapters 5 and 6 deal with slenderness (or lack thereof) of rings and modules, and in particular rings of functions. I have made an effort to follow aesthetic principles in presenting beautiful results with beautiful proofs and if such were not available, additional effort had to be made to come closer to that ideal.

I have included the introductory chapter to establish terminology and summarize the main results used in the text, but have also introduced some challenges, even in the introduction, for the reader who would like to be challenged right away. The Appendix introduces the reader to a minimum of set theory needed, in particular to the (non-)measurable cardinals. My effort to make the monograph self-contained, clearly, has to be limited. I have thus assumed that the reader has acquired fundamental notions of topology, algebra, and other topics usually taught at undergraduate level. One consequence is that definitions of numerous kinds of rings were not given in view of the fact that such definitions and basic properties may be obtained from a wide range of sources. A number of results that are auxiliary to this text are, owing to limitations of space, stated without proof, and the reader is referred to exact references for further insight into those results. For the most part, the terminology, notions, and ideas are presented in a linear fashion; there are however several exceptions when a notion is mentioned before it is defined. The indexes of notation, of names, and of various subjects then come in handy to guide the reader to the right place(s). I have made the indexes as detailed as I could and I made an effort to make them useful to the reader beyond the generic and automatic indexing software possibilities.

Statements, such as Theorems, Propositions, Lemmas, Definitions, Notes, Remarks, and Examples, are numbered according to their chapters and the order they assume in that chapter. Thus, Theorem 1.26 indicates that it is in Chapter 1, statement number 26 in order. Every chapter ends with a set of exercises of relatively increasing difficulty, totaling over 350. Exercises likewise bear their chapter label first so that Exercise 2.10 is in the second chapter, 10th in order. The references are given for harder exercises where the reader can consult the original sources. Topics of the exercises often merit expansion, but those options are left for Volume II. Each chapter also has a section with problems (about 130 in total); the answers to these problems are not necessarily known to the author and the problems listed may provide research topics for inquisitive minds. The intentional ambiguity with some of the problems is meant to encourage the reader to let his imagination lead him into related areas. Trying difficult problems is beneficial, for even if we do not solve them we learn much and reach into unexplored areas by simply making an effort to solve them.

I have striven to give accurate references that I hope go to the original sources of the results, as much as possible. This led to sources unjustly buried in dissertations and less-advertised publications. A number of results are published for the first time in this monograph. Many others are improved versions of known results; in the latter case, the inspiration sources are given. In the effort to trace the development of ideas to the earliest sources, I may not have succeeded fully, at least in some instances. In addition, I have tried to follow the path of ideas by tracing possible anticipators and predecessors to subsequent results. One of the reasons is self-utilitarian: just as Orion carried his servant Kedalion on his shoulders so have I stood on the shoulders of giant masters. Thus the historical notes every chapter concludes with may be viewed as the starting point for more in-depth historical research.

I would like to thank László Fuchs for directing my attention to slenderness and helping to get me started, at a time when I wanted to pursue other mathematical research. John Dauns (who, sadly, passed away) and Christian U. Jensen were a source of perpetual encouragement. I benefited from set-theoretic discussions (which by far exceeded the set-theoretical scope of this volume) with James Cummings, Paul Eklof, Joel Hamkins, Dana Scott, and Robert Solovay, who was my good host at UC Berkeley. I owe special gratitude to George Bergman, who read drafts of Chapter 3, the Appendix, and excerpts from the Introduction and Chapter 2, and who made numerous useful

comments along the way. Christian Jensen had access to drafts of Chapters 1, 2, and parts of Chapter 3. Special thanks go to participants and contributors in my Berkeley algebra seminars, particularly to Mark Davis, Greg Marks, and Jonathan Farley. Ivko Dimitric, as usual, spotted places that needed improvement; his guidance with section 1.6 is invaluable. I have read an enormous number of papers, some only tangentially related to this work, and this would not have been possible without access to numerous libraries from Stanford University to City University of New York. Mrs. Heather Eva of the University of Exeter Library found numerous papers and copied them for me, in the early stages of this work. Jan Okninski was helpful in sending me some Polish references, and Alexander A. Mikhalev helped with some Russian names. Finally my thanks go to the staff of Cambridge University Press, especially Roger Astley, who exhibited heroic patience while waiting for my manuscript to be available for print, and Clare Dennison, who joined in for good measure. The publisher's TeX support deserves all the accolades and so does the copy editor who spotted many points that needed correction. I implemented most of her useful suggestions, except that I did not remove numerous commas in the text, because I believe that a written word should reflect the way a writer speaks.

I do not claim this book to be perfect (in spite of considerable improvements effected through suggestions of colleagues), rather, it is only almost perfect; I believe, its distance from perfection is about 0.141592653589... In addition to the inherent imperfections I am not aware of, I have intentionally left a few points that need bettering, for good luck. The esteemed reader is encouraged to seek all these points needing improvement and contact me with any sort of feedback he may have; the book may only benefit from it.

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