Geometry of String Theory Compactifications

String theory is a leading candidate for the unification of universal forces and matter, and one of its most striking predictions is the existence of small additional dimensions that have escaped detection so far. This book focuses on the geometry of these dimensions, beginning with the basics of the theory, the mathematical properties of spinors, and differential geometry. It further explores advanced techniques at the core of current research, such as G-structures and generalized complex geometry. Many significant classes of solutions to the theory's equations are studied in detail, from special holonomy and Sasaki–Einstein manifolds to their more recent generalizations involving fluxes for form fields. Various explicit examples are discussed, of interest to graduates and researchers.

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CAMBRIDGE
UNIVERSITY PRESS

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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108473736
DOI: 10.1017/9781108635745

First published 2022

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Tomasiello, Alessandro, 1974- author.
Title: Geometry of string theory compactifications / Alessandro Tomasiello.
Description: Cambridge : Cambridge University Press, 2021. | Includes
bibliographical references and index.
Identifiers: LCCN 2021029912 (print) | LCCN 2021029913 (ebook) |
ISBN 9781108473736 (hardback) | ISBN 9781108635745 (epub)
Subjects: LCSH: String models–Mathematics.
Classification: LCC QC794.6.S85 T66 2021 (print) | LCC QC794.6.S85
(ebook) | DDC 539.7/258–dc23
LC record available at https://lccn.loc.gov/2021029912
LC ebook record available at https://lccn.loc.gov/2021029913

ISBN 978-1-108-47373-6 Hardback

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Cambridge University Press
978-1-108-47373-6 — Geometry of String Theory Compactifications
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Preface

There are already several excellent references on string theory [1–11]. This book focuses on one particular aspect: the geometry of the extra dimensions. Many interesting techniques have been developed over the years to find and classify string theory vacuum solutions, such as G-structures and pure spinors; I felt it would be useful to collect these ideas in a single place.

The intended audience is mostly advanced graduate students, but I tried to make the book interesting also to more experienced researchers who are not already working on this subject. I assume the reader has basic knowledge of general relativity, Lie groups, and algebras, and nodding acquaintance of the main ideas of supersymmetry. Proficiency in quantum field theory is very welcome but not heavily used. The basics of string theory and supergravity are recalled in Chapter 1, but in a presentation skewed toward the needs of the rest of the book, and not meant to give a complete picture of the field. Many details, such as the supersymmetry transformations, are postponed to a later stage, after a long mathematical detour in Chapters 2–7 allows us to present them with the appropriate level of sophistication.

Chapters 2 and 3 focus on the algebraic properties of spinors, and their deep relationship with forms. Here spacetime is taken to be flat. With respect to other introductions to these topics, I have emphasized the relation between a spinor and its bilinear tensors, and reviewed how forms can be considered as spinors for a doubled Clifford algebra; these are central to efforts in later chapters to rewrite supersymmetry in terms of exterior algebra. I have also considered a wide range of dimensions, both in Lorentzian and Euclidean signatures; this is perhaps a bit more than is really needed in later chapters, but it might be useful for readers who intend to go beyond the topics covered in the book. These chapters are also the most technically detailed; the aim was to teach how to carry out these computations as painlessly as possible – I have taken care to describe most steps, without resorting too often to the magical sentence "it can be shown that" to hide inaccessible derivations.

Chapters 4–7 are dedicated to geometry, but I have tried to keep them focused on physics needs. Chapter 4 is an introduction to differential geometry; these are standard topics, but sometimes they give an occasion to put the techniques of the earlier chapters to good use. In Chapter 5, we encounter G-structures, a well-known geometrical concept that has become very useful in supersymmetry. It is a very general framework, and I have discussed complex, Kähler, and Calabi–Yau geometry from this point of view. Kähler manifolds are those where computations are easiest, and so the entire Chapter 6 is devoted to them. Chapter 7 is devoted to manifolds with special holonomy where the Ricci tensor vanishes, such as Calabi–Yau's. This includes a lengthy close-up on conical manifolds, which are important later for AdS compactifications.

We get back to physics with Chapter 8. This is an elementary introduction to compactifications with relatively little mathematics, in the simple settings of pure gravity and string theory without flux fields. I have also provided here a quick review of four-dimensional supergravity, for later use. Chapter 9 starts with Calabi–Yau compactifications; these are not too realistic but are still the field's gold standard for rigor and depth. We later modify them by including D-branes and fluxes, covering in particular the important F-theory and conformal Kähler classes of Minkowski vacua.

Chapter 10 is a more systematic investigation of vacuum solutions. Here we finally introduce the supersymmetry transformation in full generality, and rewrite them in terms of forms using G-structures, and more precisely their *doubled* variants using pure forms from Chapter 3. This chapter is again rather technical at times, but the result is a very general system of form equations, which we can then use to look for supersymmetric solutions without having to consider spinors any more. Here and elsewhere, parts marked by an asterisk are harder and can be skipped on first reading. At the end of the chapter, we give a geometrical interpretation to this system in terms of so-called generalized complex geometry. We then proceed in Chapter 11 to a more detailed review of AdS_d solutions in $d \geq 4$, focusing on supersymmetric ones but mentioning supersymmetry-breaking in various instances. In some cases, we can give a complete classification of explicit solutions. We end in Chapter 12 with a quicker review of efforts to obtain dS vacua and of the swampland program, and with some final thoughts.

The book is not meant to be comprehensive. Notably, I have mostly focused on vacuum solutions, perhaps not paying enough attention to the broader geometry of reductions. After Chapter 1, I have devoted most attention to type II supergravity, and perhaps not enough to M-theory and heterotic strings. In general, I almost always avoided $d < 4$ vacua; and I have covered holography only superficially, in Chapter 11. Several other important topics have not been given the space they deserved. I hope readers disappointed by such omissions will forgive me after checking the total page count, which is already testing my editors' patience; I was also wary of the danger of producing a soulless encyclopedia. In general, the number of pages dedicated to a subject should not be construed as a judgment of its importance. I have been lengthier on topics that I feel are less thoroughly covered in other books, and sketchier on those where lots of great material is available already, and which I am including for context and completeness. On controversial issues, I have tried to represent all sides as fairly as I could; I have not tried to hide my opinion, but I believe the proper place to articulate it is in research articles.

I have learned these topics from my teachers, my collaborators, and my students. I am especially grateful to Loriano Bonora, Michael Douglas, Davide Gaiotto, Mariana Graña, Shamit Kachru, Dario Martelli, Ruben Minasian, Michela Petrini, and Alberto Zaffaroni. During the writing phase, I was helped by Bruno De Luca, Suvendu Giri, Andrea Legramandi, Gabriele Lo Monaco, Luca Martucci, Achilleas Passias, Vivek Saxena, and Riccardo Villa, and by the great staff at CUP. Special thanks go to Francesca Baviera, Concetta Fratantonio, and Luciano Tomasiello, although I am sorry the latter could not wait to see this finished. In spite of all this help, I am of course aware that the final product will turn out to have lots of typos, imprecisions, and outright mistakes; I will maintain a list of corrections on my personal website.

Conventions

- Lorentzian signature is "mostly plus."
- The word "generic" means "for any choice except for a set of measure zero."
- The antisymmetrizer of k indices is denoted by square brackets and includes a $1/k!$; for example, $v_{[m}w_{n]} = \frac{1}{2}(v_m w_n - v_n w_m)$. The symmetrizer is denoted by round brackets, so $v_{(m}w_{n)} = \frac{1}{2}(v_m w_n + v_n w_m)$. A vertical slash | is used to exclude indices from these operations.
- The floor function $\lfloor x \rfloor$ is the integer part of x or, in other words, the largest integer n such that $n \leq x$.
- The chiral matrix is $\gamma = c\gamma^0 \ldots \gamma^{d-1}$ ($c\gamma^1 \ldots \gamma^d$) for Lorentzian (Euclidean) signature. The constant c is constrained by (2.1.20) so that $\gamma^2 = 1$: notably, we take $c = \mathrm{i}$ for $d = 4$ Lorentzian, $c = -\mathrm{i}$ for $d = 6$ Euclidean, $c = 1$ for $d = 10$ Lorentzian.
- The identity matrix in d dimensions is denoted by 1_d or often simply by 1.
- d is most often the real dimension of a manifold; occasionally it denotes degree of a polynomial. Complex dimension is sometimes denoted by N.
- When working in an index-free notation, we use the same symbol v for a vector field with components v^μ and its associated one-form $g_{\mu\nu}v^\nu$.
- Indices $\mu, \nu \ldots$ are for Lorentzian signature; $m, n, \ldots$ for Euclidean signature; $i, j \ldots$ are holomorphic indices. An exception is $d = 10$ (and $d = 26$) Lorentzian, where we use $M, N \ldots$. Flat (vielbein) indices are $a, b \ldots$. Indices $\alpha, \beta \ldots$ are usually spinorial.
- The vielbein $e^a = e^a_m \mathrm{d}x^m$ is an orthonormal basis of one-forms, so the line element is $\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m \mathrm{d}x^n = e^a e_a$. Its inverse is denoted by $E_a = E_a^m \partial_m$, an orthonormal basis of vector fields. We also often need a holomorphic vielbein, defined by (5.1.22), (5.1.35), and hence $\mathrm{d}s^2 = \sum_{a=1}^{d/2} h^a \bar{h}^{\bar{a}}$.
- The spinor covariant derivative is $D_m = \partial_m + \frac{1}{4}\omega_m^{ab}\gamma_{ab}$ (Section 4.3.3).
- The components of a k-form are defined by $\alpha_k = \frac{1}{k!}\alpha_{m_1 \ldots m_k}\mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_k}$. The Clifford map associates to it a bispinor $\alpha\!\!\!/_k \equiv \frac{1}{k!}\alpha_{m_1 \ldots m_k}\gamma^{m_1 \ldots m_k}$, with $\gamma^{m_1 \ldots m_k} \equiv \gamma^{[m_1} \ldots \gamma^{m_k]}$. For lengthy expressions, we also use the notation $(\alpha_k)_/ \equiv \alpha\!\!\!/_k$, but often we don't use any symbol at all and denote by α_k both a form and the associated bispinor, with an abuse of language.
- A vector field acts on a spinor as $v \cdot \eta \equiv v^\mu \gamma_\mu \eta = v_\mu \gamma^\mu \eta = v\!\!\!/\eta$, or also just $v\eta$ by the previous point.
- $\mathrm{vol} = e^1 \wedge \ldots \wedge e^d$ is the volume form, while the volume of a manifold M is denoted by $\mathrm{Vol}(M)$.
- A chiral spinor η_+ is said to be pure if it is annihilated by $d/2$ gamma matrices; in flat space, this defines a notion of a (anti-)holomorphic index, for which we use the convention $\gamma_{\bar{\imath}}\eta_+ = 0$

- The complex conjugate of a complex number z is denoted by z^* or $\bar{z}$. For a complex matrix M, $M^\dagger \equiv (M^*)^t$. The conjugate of a spinor is $\zeta^c \equiv B\zeta^*$ (Section 2.3.1). In Lorentzian signature, we also define $\overline{\zeta} = \zeta^\dagger \gamma_0$.
- We typically use the letter ζ for spinors in Lorentzian signature, and η for Euclidean signature.
- Span$\{v_1, \ldots, v_k\}$ is the vector subspace of linear combinations of the v_a.

Introduction

The idea that spacetime might have additional dimensions might seem preposterous at first. It has come to the fore of current research in theoretical physics by two strands of thought.

0.1 String theory

One comes from attempts at quantizing gravity. The problems one encounters in general relativity at high energies suggest that it is superseded in that regime by a different theory. A prominent candidate is *string theory* (to be reviewed in Chapter 1). It describes interacting strings, which at low energies are seen as particles, some of which behave as gravitons, thus reducing at low energies to a version of general relativity (GR). The other particles behave in ways that look complicated enough to accommodate the phenomena we see in particle physics. So string theory solves the high-energy problems of GR, and gives a possible strategy to unify not only all forces but also all matter. Remarkably, the theory is so constrained by various anomalies that it has no free parameters. This is perhaps what one should expect from a unified theory of all physics.

String theory does come, however, with a heavy conceptual framework. This includes supersymmetry, which plays an important role in the theory's internal consistency; it could be broken spontaneously at Planck energies and be hidden from observations for a long time. More importantly for us, string theory only works in more than four dimensions. In its best-understood phase, *six* additional dimensions are needed, with a seventh also sometimes emerging. To avoid conflict with observations, we need to postulate that the compact space M_6 they span is small enough that current experiments have not revealed it yet. A *compactification* is a spacetime that looks four-dimensional macroscopically, even if it actually has a larger number of dimensions.

0.2 Kaluza–Klein reduction

This idea is natural enough that it had been considered long before string theory [12, 13]. The reason is that it gives a simpler, independent way to unify gravity with other elementary forces. This was first noticed in GR with a single additional dimension, wound up on a circle S^1 with an extra coordinate x^4. The various components of the metric are viewed by a four-dimensional observer as fields of different spin: $g_{\mu\nu}$ as a four-dimensional metric, $g_{\mu 4}$ as a vector field, and g_{44} as a scalar. The first two can be interpreted as describing gravity and electromagnetism in four dimensions.

The field dependence on the extra dimension x^4 also gives rise to a "tower" of massive spin-two fields with masses

$$m_k = \frac{2\pi k}{L}\,, \tag{0.2.1}$$

where L is the size of the S^1. A similar phenomenon can be seen already with a free scalar σ on $\mathbb{R}^4 \times S^1$: if we expand σ in a Fourier series with respect to x^4 and plug the expansion in the Klein–Gordon equation $(\partial_\mu \partial^\mu + \partial_4^2)\sigma = 0$, the term ∂_4^2 gives a mass (0.2.1) to the kth Fourier mode. With gravity, $g_{\mu\nu}$, $g_{\mu 4}$, and g_{44} all undergo the same phenomenon: the massive spin-two fields then "eat" the massive modes of the other components, in a version of the Brout–Englert–Higgs (BEH) mechanism. The infinite sequence of masses (0.2.1) is called a *KK tower*, and the corresponding fields are called *KK modes*.

As expected by dimensional analysis, (0.2.1) are inversely proportional to L; so when L is small, these masses are large and might have avoided detection so far. Even $L \sim 10^{-19} m$ leads to $m_k \sim O(\text{TeV})$. We will review in more depth the physics of this five-dimensional model in Section 8.1.1.

With $d > 1$ additional dimensions, one can consider more complicated spaces M_d, which can now also realize Yang–Mills (YM) theories. The symmetry group of M_d becomes the YM gauge group. From this point of view, the idea of extra dimensions is an evolution of that of "internal symmetry" in the world of elementary particles. It is the postulate that those symmetries have a geometrical origin.

0.3 String compactifications

The topic of this book is the study of the "internal space" of string theory. While the theory itself has no free parameters, the choice of M_6 introduces a lot of freedom. As we will see in Section 4.2.5, the possible topologies for a six-dimensional compact space are classified by a few algebraic data (the dimensions of two vector spaces and two polynomial functions). But the space of possible metrics for each given topology is infinite dimensional.

Perhaps the simplest question we can ask is whether by compactifying on M_6 we can find at least a *vacuum solution*: namely, one where the macroscopic spacetime is empty. This means that the stress energy tensor is zero, or consists at most of a cosmological constant. Such a space should locally have as many symmetries as flat Minkowski space, and is said to be *maximally symmetric (MS)* (Section 4.5); the possibilities are Minkowski space itself, de Sitter (dS) space, and anti-de Sitter (AdS) space, with a positive, zero, or negative cosmological constant Λ. We will argue in Chapter 8 that for such a solution the line element for the ten-dimensional metric reads

$$\mathrm{d}s^2_{10} = \mathrm{e}^{2A}\mathrm{d}s^2_{\mathrm{MS}_4} + \mathrm{d}s^2_{M_6}\,. \tag{0.3.1}$$

$A = A(y)$ is called the *warping function* of the coordinates y^m on M_6, and this form of the metric is called *warped product*.

The question of which M_6 lead to vacuum solutions is already rather hard: it involves solving the equations of motion, which reduce to partial differential equations on M_6. An easy case is when all of string theory's fields have zero expectation value except the metric; the equations of motion then say that $R_{MN} = 0$. These imply that the maximally symmetric $\mathrm{MS}_4 = \mathrm{Mink}_4$, and that M_6 is Ricci-flat (Chapter 7). Many such spaces are the so-called *Calabi–Yau manifolds*. When the other fields are also present, finding and classifying solutions is much harder. This is one of the main topics of this book.

After finding a vacuum solution, one would like a description of the physics one would observe in it. Here one faces a choice between precision and broadness. At one extreme, the *KK spectrum* is the information about all the particle masses and spins for a single given vacuum, but without any information about their interactions.

At the other extreme, one focuses on a small subset of fields, but with complete information about their interactions, and in particular about the potential for the scalars. Such an action S_4 might describe many vacua at once. As usual, we call it an effective theory if the scalars we kept span a "valley" with a relatively mild potential V_{eff}, much smaller than the potential for the scalars we discarded. Sometimes in string theory we are forced to work without such a scale separation; one calls this a *nonlinear reduction*, and we then want at least that its vacua correspond to vacuum solutions of fully fledged string theory. We will give a longer introduction to these ideas in Section 8.3, and then see several examples in later parts of the book.

Sometimes one reverses this procedure, and one uses an effective theory or nonlinear reduction to find new vacuum solutions rather than to describe the physics of one that was previously found. Some of the theory's most celebrated solutions have been found this way. However, our focus will be on techniques to find vacua directly in ten (or eleven) dimensions.

0.4 Supersymmetric vacua and geometry

To find vacuum solutions, it proves easier to start from those where supersymmetry is partially preserved and to break it later, rather than trying to solve string theory's equations of motion in general. Technically, preserved supersymmetry gives a first-order system of equations, which partially implies the second-order equations of motion. More importantly, supersymmetry helps finding solutions because it naturally invokes several deep geometrical ideas.

We cannot cover the internal M_6 with a single coordinate system; we have to use several, related by coordinate changes called transition functions, in general valued in the group $\mathrm{GL}(6, \mathbb{R})$ of invertible matrices. A *G-structure* on M_6 is a choice of transition functions valued in a smaller group G (Chapter 5). The infinitesimal parameters for supersymmetry are spinors η; they naturally define a G-structure, with $G = \mathrm{Stab}(\eta)$ their little group (or *stabilizer*), the group of rotations that keep η invariant. This helps trading η with other, nonspinorial geometrical objects on M_6; often antisymmetric tensors, or *forms*.

For example, a single η on M_6 defines an SU(3)-structure, which can also be defined by the metric and a complex three-form (an antisymmetric tensor with three indices) Ω. It is further possible to trade even the metric for a real two-form J. The G-structure techniques allow one to recast the supersymmetry equations, originally involving η, directly in terms of J and Ω. Often one then recognizes a well-known mathematical concept, and this helps finding solutions. This procedure is also natural because most of the string theory fields beyond the metric are themselves forms, analogues with many indices of the electromagnetic field-strength $F_{\mu\nu}$.

Some of the most interesting vacua are in type II, where there are two η^a. In this case, it proves more fruitful to work with a doubled, or *generalized*, version of the rotation group. Again, the upshot is that we may trade the data of the η^a and of the metric with forms, this time a pair $\Phi_\pm$ of them with an algebraic property called *purity* (Chapters 2 and 3). In this language, the supersymmetry equations become particularly elegant, making contact with *generalized complex geometry* (Chapter 10).

The fact that the metric is included in this trade-off with forms is particularly intriguing. It is reminiscent of previous attempts to reformulate GR such as [14, 15].

0.5 The cosmological constant

The observed cosmological constant is positive, so one would like to focus on the case $\mathrm{MS}_4 = \mathrm{dS}_4$. These are actually the hardest solutions to obtain. To see why, consider a general gravitational theory with an Einstein–Hilbert (EH) kinetic term [16–18]. The equations of motion read $R_{MN} - \frac{1}{2} g_{MN} R = 8\pi G_{\mathrm{N}} T_{MN}$, where as usual G_{N} is Newton's constant, R_{MN} is the Ricci tensor, R its trace, and T_{MN} the stress–energy tensor of various matter fields; the indices $M, N = 0, \ldots, 3 + d$. The "trace-reversed" Einstein equations are

$$R_{MN} = 8\pi G_{\mathrm{N}} \left(T_{MN} - \frac{1}{2+d} g_{MN} T_P{}^P \right) . \tag{0.5.1}$$

For a warped product metric as in (0.3.1) (generalizing $M_6 \to M_d$),

$$R_{\mu\nu} = \left(\Lambda - \frac{1}{4} \mathrm{e}^{-2A} \nabla^2 \mathrm{e}^{4A} \right) g^4_{\mu\nu} , \tag{0.5.2}$$

where $g^4_{\mu\nu}$ are the components of the metric of the external MS_4 space, Λ its cosmological constant (normalized as $R^4_{\mu\nu} = \Lambda g^4_{\mu\nu}$), and ∇ the internal covariant derivative. (We will derive (0.5.2) in an exercise in Chapter 4.) In coordinates where $g^4_{00} = -1$, the time components of (0.5.1) give

$$-\mathrm{e}^{2A}\Lambda + \frac{1}{4}\nabla^2 \mathrm{e}^{4A} = 8\pi G_{\mathrm{N}} \mathrm{e}^{2A} \left(T_{00} - \frac{1}{2+d} g_{00} T_P{}^P \right) . \tag{0.5.3}$$

The parenthesis on the right-hand side is nonnegative if the higher-dimensional theory obeys the *strong energy condition*. This is an assumption often made in general relativity, for example in proving singularity theorems for black holes and cosmology; see, for example, the discussions in [19, 4.3;8.2] and [20, chap. 9].

Supposing it holds, we integrate (0.5.3) on the compact M_6; the second term on the left-hand side is a total derivative and gives no contribution, so

$$\Lambda \leq 0\,. \tag{0.5.4}$$

We conclude that a theory with an EH kinetic term obeying the strong energy condition has *no de Sitter compactifications*. Even Minkowski compactifications are only marginally allowed, requiring the parenthesis in (0.5.3) to vanish.

Does this apply to string theory? We mentioned that it reduces at low energies to a version of GR, which due to supersymmetry is called *supergravity*. In this regime, the graviton has an EH kinetic term, coupled to various other fields and to certain localized sources. All the fields satisfy the strong energy condition, except for a term called *Romans mass*. But this possible loophole in the argument was closed in [18] (Section 10.3.1).

For localized sources, violating the strong energy condition requires negative tension, leading to repulsive gravity, and usually to instabilities for dynamical objects. The sources in string theory are of two types, called *D-branes* and *O-planes* (Sections 1.3 and 1.4.4). The first are defined as spacetime defects on which strings can end; this makes them dynamical. The second arise after quotienting string theory by a parity-like symmetry, and arise at its fixed loci; so they are not dynamical. Some of them have indeed negative tension, and so they do invalidate (0.5.3).

The conclusion is that, in the regime where string theory is described by supergravity, de Sitter compactifications require O-planes. Minkowski compactifications also need them, unless all the fields except gravity are turned off.

0.6 Beyond supergravity

Let us consider now a $d = 10$ effective field theory S_{10}. This is useful in a regime of energies high enough to see the extra dimensions, but low enough to see strings as particles; it is not to be confused with the $d = 4$ effective action S_4, relevant at lower energies where we cannot resolve the extra dimensions. In S_{10}, supergravity is the collection of the most relevant operators at energies well below the Planck scale, which in turn is related to a new fundamental length scale l_s, the "typical length" of strings. But supergravity is not renormalizable; this manifests itself in the presence of higher-derivative corrections, which become relevant at high energies, or when the curvature gets large. These are not known completely, but there is no reason to expect that the result (0.5.4) still holds when they are introduced. For example, one famous leading correction has the form $\int \mathrm{d}^{10}x(\text{Riemann})^4$. So not even the metric appears simply through an EH kinetic term, as in supergravity. Unfortunately, only the first few terms have been computed.

This introduces other challenges. These corrections will contribute to the potential of the $d = 4$ effective action S_4 (Chapter 8), whose vacua should approximate string theory's vacuum solutions as defined earlier. Very naively, suppressing the dependence on other fields, we will see that the EH term and the $(\text{Riemann})^4$ term give two contributions to the potential:

$$V_4 \sim ar^{-2} + br^{-8} , \tag{0.6.1}$$

where $r \equiv R/l_s$ is the length scale of the internal space M_6, in units of the string length. If a and b have opposite sign, this has an extremum at $r = (-4b/a)^{1/6}$; but if $a, b \sim O(1)$, also $r \sim O(1)$, and $R \sim l_s$. But in this regime, other (Riemann)^k, $k > 4$ corrections would also be relevant, contributing further terms r^{-2k} to (0.6.1); so we cannot trust our extremum. This illustrates a general issue: if we find a solution by using one string correction to supergravity, we can expect it to be in a regime where *all* string corrections are relevant, where we in fact cannot compute anything. This is the *Dine–Seiberg problem* [21].

Fortunately, not all the terms in the $d = 4$ effective potential are $\sim O(1)$. The aforementioned form fields have to satisfy a certain Dirac quantization; this introduces integers, which can be taken to be large, introducing a hierarchy that eventually makes $R \gg l_s$. From the point of view of S_4, this is behind the existence of most vacua, but usually the terms that compete originate from the leading supergravity approximation. (For the Calabi–Yau vacua, the leading supergravity contribution vanishes; we will see in Chapter 9 a more delicate argument to show that the corrections don't destroy the vacuum.)

0.7 Overview of vacua

The argument (0.5.4) indicates that finding vacua is easiest when the cosmological constant is negative, which as we mentioned is contrary to observations. These AdS vacua have found applications in *holography*, which relates them to quantum field theory models with conformal invariance, or conformal field theories (CFTs). Another reason not to discard them is that the supersymmetry-breaking procedure sometimes also changes Λ. For these reasons, we dedicate Chapter 11 to a survey of such vacua. Several classification results are available here, and several more are likely to emerge in the near future, as techniques improve. For example, a list of all supersymmetric AdS_6 and AdS_7 solutions has been achieved relatively recently.

Minkowski vacua are more tightly constrained. Relative to AdS, this is expected, if nothing else because $\Lambda = 0$ is an equality, not an inequality. A priori the general equations seem to allow for M_6 of any curvature; but so far the vast majority of known supersymmetric vacua are related to Calabi–Yau manifolds, one way or another. For example, in a famous class in Chapter 9 the Calabi–Yau metric is only modified by an overall function. Until recently, one might have thought this to be an artifact of technical limitations in including O-planes, which as we have already argued are necessary. However, many AdS vacua with O-planes have now been constructed and seem to allow for a far greater variety of internal spaces.

Finally, for the de Sitter case, the situation gets even less clear (Chapter 12). One additional complication is that supersymmetry is necessarily broken and cannot guide us any more. Most models evade (0.5.4) by involving both O-planes and quantum effects, which are harder to control. As a result, all of them have attracted some objections.

The first and most successful proposal, the KKLT model [22], again obtained by modifying a Calabi–Yau metric on M_6, generates a large quantity of dS vacua, with numbers such as 10^{hundreds} or even $10^{\text{thousands}}$ often quoted. The resulting picture has been dubbed the *string theory landscape* [23], borrowing a metaphor from protein-folding research. Leaving aside for now any criticism of this and other models, the possibility that a seemingly formidable obstacle such as the cosmological constant could be overcome so easily has suggested that the number of vacua reproducing *all* other observed features of our Universe could still be very large. This has created much confusion in causal observers of the field. Perhaps string theory has no predictive power?

This question appears misguided. First of all, the large numbers 10^N arise from discretizing an N-dimensional continuous space; in other words, from allowing several discrete possibilities to a set of N free parameters. Conceptually, this is not that different from the 19 (or more) free parameters in the standard model, which prior to experiment has an ∞^{19} of possibilities. Rightly, no one complains about the latter large number because, given enough experiments to fix the parameters within a certain range, the standard model makes testable predictions about other experiments. This would be true as well for string theory; the fact that such experiments are beyond human capabilities for the foreseeable future is of course unfortunate, but is in the nature of the problem of quantum gravity, whose characteristic scale is after all m_{Planck}.

Even more importantly, string *theory* is a framework, within which there are *models* with free parameters, such as the KKLT model. It would of course be senseless to criticize quantum field theory because it cannot predict the standard model of particle physics from first principles, or criticize quantum mechanics because it does not predict the potential in the Schrödinger equation. Of course, quantum field theory is a scientific *theory*, in the appropriate sense: given enough experimental data, it can provide a *model*, such as the Standard Model, which makes new experimental predictions.

An alternative point of view is to focus on the vacua that *cannot* be found in string theory. The *swampland* program [24] looks for models that look consistent in field theory but cannot be coupled to quantum gravity. While the inspiration often comes from string theory, the aim is to find universal properties that are valid beyond it. In recent years, this program has started to clash with many of the predictions of the effective field theory approach, including the existence of dS vacua. Some of this debate is covered in the final Chapter 12, with the unfortunate result of ending with more questions than answers.