Cambridge University Press
978-1-108-47373-6 — Geometry of String Theory Compactifications
Alessandro Tomasiello
Excerpt
More Information

1 String theory and supergravity

As stated in the Preface, this book assumes some rudimentary knowledge of string theory, but it is a good idea to recall the basics. The field is notoriously vast and complex, so this chapter should not be understood as a replacement for serious study on one of the many great introductions [1–11]. In most of the book, we will approximate string theory by *supergravity*, an effective theory of gravitons and other fields; the presentation will be biased toward that.

In this chapter, we also assume knowledge of general relativity (GR) and some acquaintance with spinors, but we will try to keep mathematical sophistication at a minimum. We will develop some ideas, such as spinors and differential geometry, in much greater detail in the next few chapters before we return to physics. Still, already in this chapter we will pepper our presentation with occasional forward references to those mathematically more advanced treatments, to whet the reader's appetite.

1.1 Perturbative strings

A quantum field collects creation (and annihilation) operators for a representation of the Poincaré group. Once one fixes the value of the momentum p of the created state, the remaining degrees of freedom are a representation of the *little group*, or *stabilizer*, of p, namely the subgroup $\mathrm{Stab}(p) \subset \mathrm{SO}(d)$ of elements that leave p invariant. This is

$$\mathrm{Stab}(p) = \mathrm{SO}(d-1)\ (p^2 < 0)\,, \qquad \mathrm{Stab}(p) = \mathrm{SO}(d-2) \ltimes \mathbb{R}^{d-2}\ (p^2 = 0)\,, \quad (1.1.1)$$

for the massive or massless case. (We will review the $p^2 = 0$ case in Section 3.3.6.) In the massless case, we would also have the possibility of selecting an infinite-dimensional representation, but this is usually regarded as exotic; so we select a finite-dimensional representation, ignoring the $\mathbb{R}^{d-2}$ factor. Ordinary fields then represent objects with finitely many degrees of freedom, which we call spin and helicity for $m^2 > 0$ and $= 0$, respectively. Moreover, we usually take these objects to interact via terms of the type $\int \mathrm{d}^d x \phi_1(x) \ldots \phi_2(x)$: these allow the value of a field to influence directly that of another only at the same point.

All these reasons make us think of the quanta of a field as point particles. To describe a quantum theory of interacting extended objects, we need to change this picture somehow. First of all, a string can have infinitely many vibration modes, so a field that creates a string must be somehow a collection of infinitely many ordinary fields. Second, extended objects can interact when their centers of mass are not superimposed. So the interaction terms should be nonlocal.

Such a *string field theory* (SFT) is fascinating but also just as complicated as our description suggests. So in fact most studies of interacting strings focus on an approach that is *first-quantized*: one first decides the Feynman diagram one wants to consider, and then computes the amplitude associated with it. (A similar approach is used sometimes in quantum field theory too, under the name of *world-line formalism*.)

In this section, we will review quickly some aspects of this perturbative treatment of string interactions. There are five possible consistent string models:

- Type IIA
- Type IIB
- Heterotic with gauge group $E_8 \times E_8$
- Heterotic with gauge group SO(32)
- Type I

All these select $d = 10$ as spacetime dimension, in a sense we will clarify later in this chapter. The last case, type I, can be viewed as a certain quotient procedure from IIB strings, which we will introduce in Section 1.4.4. So in this section we will discuss the other four. We will actually start our discussion from a model that has a *tachyon*, namely a scalar with a negative mass, but whose discussion is simpler: the *bosonic* string.

1.1.1 Bosonic strings

The action for a particle moving in a curved background is proportional to its "length in spacetime," namely, to the proper time measured along its world-line (its trajectory γ in spacetime):

$$S_{\text{part}} = -m \int_\gamma \mathrm{d}\sigma_0 \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \qquad (1.1.2)$$

where $x^\mu(\sigma_0)$ are the coordinates of the point in spacetime as a function of the world-line coordinate σ_0, and $\dot{x}^\mu \equiv \partial_0 x^\mu$. In flat space, this is indeed minimized on straight lines in spacetime, which maximize proper time. For curved $g_{\mu\nu}$, (1.1.2) is minimized on geodesics. If we also have a Maxwell field and our particle is charged, we have to add a term

$$S_{\text{part,EM}} = q \int_\gamma \mathrm{d}\sigma_0 A_\mu \partial_0 x^\mu, \qquad (1.1.3)$$

where q is the charge, and A_μ is the vector potential. In Section 4.1.4, we will see that the integrand is an example of a natural operation called *pull-back*.

String action

By analogy with (1.1.2), the natural action for a string would seem to be the volume of its two-dimensional world-sheet in spacetime. However, it is classically equivalent to the *Polyakov action*, which is easier to quantize:

$$S_{\text{F1},g} = -\frac{1}{2} T_{\text{F1}} \int_\Sigma \mathrm{d}^2\sigma h^{\alpha\beta}\sqrt{-h} g_{MN} \partial_\alpha x^M \partial_\beta x^N . \qquad (1.1.4)$$

This type of action is also called a *sigma model*, for reasons going back to four-dimensional models of mesons, or sometimes *nonlinear* sigma model when g_{MN} is not flat. The $x^M(\sigma_0, \sigma_1)$, $M = 0, \ldots, d-1$, describe the embedding of Σ in physical spacetime (often called *target space*), and h is a metric on Σ. The mass m in (1.1.2) has been replaced by the mass/length ratio, or *tension*:

$$T_{\rm F1} = \frac{1}{2\pi l_s^2}\,. \tag{1.1.5}$$

"F" stands for *fundamental*, to distinguish this string from other extended objects that will appear later; 1 denotes the space extension of the string. The constant l_s is called *string length*. (We will always keep it explicit in this chapter, but later we will often work in string units and set $l_s = 1$.)

In this section, we are going to focus on strings that are *closed* or, in other words, that have no boundary. A generic[1] time slice is then a collection of several copies of the circle S^1. The time evolution of each of these for a finite time will be a cylinder; then σ^1 is a periodic coordinate, $\sigma^1 \sim \sigma^1 + \pi$. These cylinders are then glued together at some values of σ^0 to obtain a general Σ.

Spectrum in flat space

Quantizing (1.1.4) is challenging for general g_{MN} but relatively easy in Minkowski space $g_{MN} = \eta_{MN}$: superficially (1.1.4) then becomes a collection of free bosons, with equations of motion $\partial^2 x^M = 0$. For a closed string, the slice at $\sigma^0 =$ constant is an S^1; there are then discrete Fourier modes for each x^M. Since the equation of motion is of second order, the states are in correspondence to the values of these Fourier modes and their derivatives. Alternatively, we can write a solution of the world-sheet equations of motion as $x^M = x^M(\sigma^+) + x^M(\sigma^-)$, where $\sigma^\pm = \sigma^1 \pm \sigma^0$, and introduce Fourier modes α_i^M, $\tilde\alpha_i^M$ for the left- and right-movers $x^M(\sigma^\pm)$. The only subtlety is that the world-sheet metric $h_{\alpha\beta}$ is a Lagrange multiplier, which gives a constraint. This can be taken care of in many ways: by solving the constraint, or by introducing Faddeev–Popov ghosts and the Becchi–Rouet–Stora–Tyutin (BRST) method (the so-called covariant quantization). Skipping many interesting details, here we will just give the results.

Even for a fixed momentum, the spectrum has infinitely many states, of the form

$$\alpha^{N_1}_{-i_1} \ldots \alpha^{N_n}_{-i_n} \tilde\alpha^{\tilde N_1}_{-\tilde\imath_1} \ldots \tilde\alpha^{\tilde N_n}_{-\tilde\imath_n} |0\rangle, \tag{1.1.6}$$

where $|0\rangle$ is the world-sheet vacuum, and i_k, $j_k \geq 0$ (possibly repeated). As we mentioned, these correspond to the vibration modes of the string, and in a spacetime picture they would require infinitely many ordinary quantum fields to create them. Their masses are

$$m^2 = \frac{4}{l_s^2}\left(\frac{2-d}{24} + N\right), \tag{1.1.7}$$

where $N = \sum i_k = \sum \tilde\imath_k$ is a nonnegative integer. The identity between these two expressions is called *level matching* and is the link between the left- and right-moving sectors, which otherwise proceed on parallel tracks. If $d > 2$, we see that the lowest

[1] The mathematical meaning of the word *generically*, which we will use in this book, is "for any choice except for a set of measure zero."

value of m^2, for $N = 0$, is actually *negative*. Such a mode is usually called a *tachyon* and signals an instability. For this reason, the bosonic string we are discussing in this section is usually only considered a toy model.

Nevertheless, it already displays a very interesting feature. For the *critical dimension* $d = 26$, the modes with $n = 1$ in (1.1.6) and (1.1.7) are massless. They read

$$\alpha^M_{-1}\tilde{\alpha}^N_{-1}|0\rangle\,, \tag{1.1.8}$$

and so they correspond to fields with two indices. Among these we thus find a *massless spin-two field* $h_{MN} = \delta g_{MN}$. The action (1.1.4) can then be thought of as a string moving in a condensate of such a field. This is a bit similar to expanding a quantum field theory (QFT) around a vacuum where a field has acquired a nonzero expectation value.

So we have found that in $d = 26$, the string modes include those that would normally be associated with a graviton. Remarkably, the scattering amplitudes one obtains with this formalism are *finite*. The string tension acts as a regulator: Taking the limit $l_s \to 0$, the scattering amplitudes become divergent again. In this limit, the theory becomes a local QFT model again, and a local theory of gravity has divergent amplitudes.

Coupling to condensates of other fields

Among (1.1.8), we find other massless modes. Following (1.1.1), we need to consider only the components of (1.1.8) in the $d - 2 = 24$ dimensions transverse to the momentum p, which are 24^2. The physical components h_{MN} of a graviton are represented by a traceless 24×24 matrix; this is the generalization of the transverse traceless (TT) gauge familiar from the treatment of gravitational waves in four dimensions. The remaining modes are thus the antisymmetric part of (1.1.8) and its trace. The fields that create these states are an antisymmetric *Kalb–Ramond* field $B_{MN} = -B_{NM}$, and a scalar field ϕ called *dilaton*. So in total the massless fields of the bosonic string are

$$g_{MN}\,, \qquad B_{MN}\,, \qquad \phi\,. \tag{1.1.9}$$

We can consider condensates of B_{MN} and ϕ, too; this leads to the extra terms in the action:

$$S_{\mathrm{F1},B,\phi} = -\frac{1}{2}T_{\mathrm{F1}}\int_\Sigma \mathrm{d}^2\sigma\left[\epsilon^{\alpha\beta}B_{MN}\partial_\alpha x^M\partial_\beta x^N + l_s^2\sqrt{-h}R_{(2)}\phi\right]. \tag{1.1.10}$$

Here $R_{(2)}$ is the scalar curvature of the world-sheet metric $h_{\alpha\beta}$, and $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The coupling with B is the natural generalization of the coupling (1.1.3). The coupling with the dilaton is peculiar in that

$$\frac{1}{4\pi}\int_\Sigma \sqrt{-h}R_{(2)} = 2 - 2g\,, \tag{1.1.11}$$

where g is the *genus* of the world-sheet Σ. This is the stringy analogue of the number of loops, and can be intuitively described (when Σ has no boundary) as the *number* *of handles*; a more formal definition will be given in Section 4.1.10. Because of this, the computation of all scattering amplitudes is organized in powers:

$$g_s^{2g-2}\,, \qquad g_s \equiv \mathrm{e}^{\phi}\,. \tag{1.1.12}$$

We can think of g_s as a *string coupling constant*: when it is small, the powers (1.1.12) are smaller for Riemann surfaces Σ of increasing g, which can be thought of as the stringy analogue of Feynman diagrams of increasing complexity.

The action

$$S_{\text{bos}} = S_{\text{F1},g} + S_{\text{F1},B,\phi} \tag{1.1.13}$$

is classically invariant under general coordinate transformation $\sigma_\alpha \to \sigma'_\alpha(\sigma_0, \sigma_1)$, if we also take care to transform the world-sheet metric $h_{\alpha\beta}$. This is a gauge invariance, in that it doesn't affect the physical configuration, the image of the world-sheet embedding $x^\mu(\sigma)$, but only how we parameterize it. Equation (1.1.13) is also invariant under Weyl rescaling $h_{\alpha\beta} \to e^f h_{\alpha\beta}$. In two dimensions, one can fix the coordinate-change freedom by taking, for example, $h_{\alpha\beta}$ to have constant scalar curvature. Even so, a residual invariance remains: coordinate transformations that leave the metric invariant up to a Weyl transformation. These are called *conformal transformations*.

Conformal invariance and effective action

It is crucial that this residual gauge invariance remains at the quantum level. It decouples potentially harmful negative-norm states that would come from the fact that x^0 in (1.1.4) has a wrong-sign kinetic term. This is similar to what happens in the quantization of the electromagnetic field, for example. Conformal invariance is also behind the absence of high-energy divergences. Usually scattering amplitudes become problematic when two particles collide at a small impact parameter. The world-sheet of a string scattering is a non-compact Riemann surface with several spikes s_i corresponding to the incoming and outgoing strings. Conformal invariance means that the distance between two points on the world-sheet has no intrinsic meaning: only ratios of distances do. So a small impact parameter might seem to correspond to two such spikes s_1 and s_2 getting close, but that only means that they are close relative to their distance from other external strings s_i. This corresponds to a Riemann surface that develops a long neck, where the two s_i are both attached, far from the others.

The Noether current associated to dilatations in a field theory is $T^{\mu\nu}x_\nu$, where $T_{\mu\nu}$ is the stress–energy tensor. This is conserved if $0 = \partial_\mu(T^{\mu\nu}x_\nu) = T^{\mu\nu}g_{\mu\nu} = T^\mu_\mu$. Evaluating the expectation value $\langle T^\mu_\mu\rangle$ of this trace is thus a way to check if there is a *Weyl anomaly*.

From the point of view of the world-sheet, the spacetime fields (1.1.9) are really couplings for the action of the fields $x^M(\sigma)$. So a Weyl anomaly can also be detected by computing the beta functions of the action (1.1.13) for the couplings (1.1.9). This can be obtained by the usual perturbative methods; the coupling for this computation is given by l_s^2, or rather the dimensionless combination $l_s^2\times$ (spacetime curvature). This results in the following three conditions:

$$R_{MN} + 2\nabla_M \partial_N \phi - \frac{1}{4} H_{MPQ} H_N{}^{PQ} + O(l_s^2) = 0\,, \tag{1.1.14a}$$

$$\nabla_M (\mathrm{e}^{-2\phi} H^M{}_{NP}) + O(l_s^2) = 0\,, \tag{1.1.14b}$$

$$\frac{2}{3l_s^2}(26-d) + R - \frac{1}{2}|H|^2 - 4\mathrm{e}^{\phi}\nabla^2 \mathrm{e}^{-\phi} + O(l_s^2) = 0\,. \tag{1.1.14c}$$

We have introduced

$$H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}\,, \qquad |H|^2 \equiv \frac{1}{6} H_{MNP} H^{MNP}\,. \tag{1.1.15}$$

This can be considered as a *field-strength* for the *potential* B_{MN}, similar to the relation between $F_{MN} = \partial_M A_N - \partial_N A_M$ and A_M in electromagnetism. Indeed, there is also a gauge transformation

$$B_{MN} \to B_{MN} + \partial_M \hat{\lambda}_N - \partial_N \hat{\lambda}_M\,, \tag{1.1.16}$$

under which (1.1.15) is invariant. The world-sheet action (1.1.10) is invariant too under this, because the transformation adds a total derivative term.

From spacetime point of view, where (1.1.9) are fields, (1.1.14) are to be interpreted as equations of motion. They can be obtained by extremizing[2]

$$S_{\mathrm{bos}} = \frac{1}{2\kappa_{\mathrm{b}}^2} \int \mathrm{d}^d x \sqrt{-g}\, \mathrm{e}^{-2\phi} \left(\frac{2}{3l_s^2}(26-d) + R + 4\partial_M \phi \partial^M \phi - \frac{1}{2}|H|^2 + O(l_s^2) \right) \tag{1.1.17}$$

with respect to (1.1.9). By dimensional reasons, κ_{b} has dimension l_s^{12}. (The metric coefficients have no dimension, while R contains two derivatives and has mass dimension two.) In general, the *Planck mass* m_{P} is defined as the mass scale entering the Einstein–Hilbert action; the *Planck length* l_{P} is its inverse, and (1.1.17) tells us that it is proportional to l_{s}.

As a consistency check, we see that flat space is a solution of (1.1.14) only if we set $d = 26$, which is the value where we found the massless fields (1.1.9) in the first place. More generally, to trust (1.1.14) we have to make sure that the expansion parameter $l_s^2 \times$ (curvature) is small, so we better solve those equations of motion separately at every order. This leads again to taking

$$d = 26\,. \tag{1.1.18}$$

It is conceptually possible to consider solutions where $d \neq 26$, and the first term in (1.1.14c) competes with the others, but in that case we have to worry that the other terms in the l_s expansion become relevant too, and we have not given them in (1.1.14). If, on the other hand, one is able to prove that a certain world-sheet model is conformal exactly, without using the l_s expansion at all, then $d = 26$ is not necessary. There are not many such cases: one is the *linear dilaton* background, where ϕ is linear in one of the coordinates. This leads to *noncritical string theories*, which historically have been important toy models.

Another point of view on the critical dimension is this. We observed that (1.1.13) has conformal invariance. Conformal transformations form a group; for flat space it is $\mathrm{SO}(d-2,2)$ for $d > 2$, but for $d = 2$ it becomes infinite dimensional. Indeed,

[2] This variation is a little more involved than the usual Einstein–Hilbert action variation because of the prefactor $\mathrm{e}^{-2\phi}$. More details will be given in Section 10.1.2.

any transformation $x^\pm \to x^{\pm\prime}(x^\pm)$ is conformal for any metric of the type $\mathrm{d}s^2 = \mathrm{e}^f \mathrm{d}x^+\mathrm{d}x^-$. The generators L_m, $m \in \mathbb{Z}$, of such transformations on the x^+ obey the Lie algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\,\delta_{n+m,0}\,, \tag{1.1.19}$$

called *Virasoro algebra*. The L_0, $L_{\pm 1}$ form an $\mathrm{SO}(1,2)$ subalgebra where c does not appear. As usual, spacetime transformations are generated by the stress–energy tensor, so these L_m are related to it. After a Wick rotation, $x_+ \to z = \sigma^1 + \mathrm{i}\sigma^0$, and we can collect all the generators in

$$T_{zz}(z) = \sum_n L_n z^{-n-2}\,. \tag{1.1.20}$$

In a Lie algebra, the commutation relations should always be linear, so we need to think of the second term in (1.1.19) as containing a new generator c, which commutes with all the others, and thus lies in the center of the algebra; so c in (1.1.19) is called *central charge*. The $\tilde{L}_m$ on the x_- variable generate a second copy of the same algebra (1.1.19), and they are collected in $T_{\bar{z}\bar{z}}$.

This c is also a measure of the Weyl anomaly: for any QFT model that is conformal on a flat (world-sheet) metric $h_{\alpha\beta} = \eta_{\alpha\beta}$, a nonzero c tells us that conformal invariance is broken for more general $h_{\alpha\beta} \neq \eta_{\alpha\beta}$. A free boson contributes $c = 1$, while the ghosts give -26. Thus if we quantize *around flat space*, where the $x^M(\sigma)$ bosons are free, for quantum conformal invariance we need to take $d = 26$.

The fact that the action (1.1.17) exists at all is nontrivial from the point of view of the world-sheet derivation we described. We can think of it as being an approximation to the string field theory action S_{SFT}, which would also contain the massive fields creating all the states (1.1.6). We can call it an effective action, in the usual quantum field theory sense: It reproduces the results one would obtain from S_{SFT}, at energies that are *low*, namely much smaller than l_s^{-1}. Indeed, another way to compute (1.1.17) is to compute string scattering amplitudes using the world-sheet approach, and then guessing what spacetime action would reproduce them.

The diagrams leading to (1.1.17) have $g = 0$ in (1.1.12), leading to $g_s^{-2} = \mathrm{e}^{-2\phi}$, thus explaining the presence of that exponential. The higher powers of l_s hidden in (1.1.17) also receive contributions from higher values of g (and thus from more complicated Feynman diagrams). So the effective action will have a double expansion in powers of both:

$$S = \sum_{j,k} S_{j,k}\, l_s^j \mathrm{e}^{k\phi}\,. \tag{1.1.21}$$

These higher-order corrections can in principle be computed; we will see some examples for superstrings. When we first discover that GR is non-renormalizable and needs (curvature)^2 counterterms [25], we might perhaps hope that by adding more and more such counterterms, with arbitrary powers (curvature)^k, we might eventually find a theory that has no divergence. Finding such a renormalizable theory of gravity would be very hard without some sort of guidance: not only would we have to find a fixed point of the renormalization group (RG) flow by going backward in energy, but we would also have to worry about modes with wrong kinetic energy, which in such theories generically abound. (Adding operators with higher numbers of derivatives to a Lagrangian also adds propagating modes, each of which might be a ghost.) String theory is renormalizable, and in principle we can reexpress it precisely as such a sum of infinitely many corrections to (1.1.17).

This discussion seems, however, to assume that the effective action is analytic in the parameters l_s, e^{ϕ}, or, in other words, that it coincides with its Taylor expansion (1.1.21). In mathematics, we know many functions that are not analytic, and they might also appear here. This is the reason we have put the word "perturbative" in the title of this section; we will make amends in Section 1.4.

Some critics of string theory complain that the theory has not been proven to be *background independent*. What they mean is that in the world-sheet approach based on (1.1.13), we first have to fix a background configuration for the spacetime fields (1.1.9), and then we can compute an action for the small fluctuations around it. A priori, it might even be unclear if this procedure is describing a single theory or a collection of theories that have nothing to do with each other. The emergence of (1.1.17) should be reassuring in this respect: that effective action can be expanded around any background, and matches the result of the world-sheet method around it. A more satisfactory rebuttal is the proof at the level of string field theory in [26].

Torus compactification

Finally, let us have a first taste of string compactifications, by supposing that the theory lives on $\mathbb{R}^{25} \times S^1$. Thus we declare one of the coordinates to be periodically identified, say $x^{25} \equiv x^{25} + 2\pi R$. Now $x^{25}(\sigma^0, \sigma^1)$ is no longer necessarily periodic as a function of σ^1, even for a closed string: rather, if we take $\sigma^1 \sim \sigma^1 + \pi$, we demand

$$x^{25}(\sigma^0, \sigma^1 + \pi) = x^{25}(\sigma^0, \sigma^1) + 2\pi w R\,. \tag{1.1.22}$$

This represents a string that winds $w \in \mathbb{Z}$ times around the S^1. Another new effect is familiar from quantum mechanics: the overall momentum of the string in the S^1 direction is now not continuous but quantized: $p^{25} = \frac{q}{R}$, $q \in \mathbb{Z}$.

The mass spectrum in $\mathbb{R}^{25}$ is now modified from (1.1.7) to

$$m^2 = \frac{4}{l_s^2}(-1 + N) + \left(\frac{q}{R} - w\frac{R}{l_s^2}\right)^2 = \frac{4}{l_s^2}(-1 + \tilde{N}) + \left(\frac{q}{R} + w\frac{R}{l_s^2}\right)^2\,, \tag{1.1.23}$$

where now $N = \sum i_k$ and $\tilde{N} = \sum_k \tilde{\imath}_k$ are no longer necessarily equal (as they were in (1.1.7)); comparing the two expressions, we have $N - \tilde{N} = wq$.

For a generic value of R, the massless spectrum is still (1.1.8) and (1.1.9); but now it should be reinterpreted. The components

$$g_{M\,25}\,, \qquad B_{M\,25} \tag{1.1.24}$$

are now two vector fields in $\mathbb{R}^{25}$; $g_{25\,25}$ is a scalar. The remaining components of (1.1.9) then give a metric, a Kalb–Ramond field, and a scalar in $\mathbb{R}^{25}$.

From (1.1.23), however, we also see another option: if $\frac{q}{R} - w\frac{R}{l_s^2} = \pm\frac{2}{l_s}$, then we have a new massless state for $N = 0$. This is possible for

$$R = l_s\,, \tag{1.1.25}$$

taking $q = -w = \pm 1$; then $\tilde{N} = \sum_k \tilde{\imath}_k = 1$. This state $\tilde{\alpha}^M_{-1}|0\rangle$ has a single index, and so it is created by a vector field. At this value of R, we also have the possibility of using the same trick with the other expression in (1.1.23), this time leading to

$q = w = \pm 1$, $\tilde{N} = 0$, $N = 1$. So we have a total of four more vector fields in $\mathbb{R}^{25}$. It turns out that these combine with the previous two (1.1.24) to give a nonabelian gauge group

$$\mathrm{SU}(2) \times \mathrm{SU}(2)\,. \tag{1.1.26}$$

This compactification was rather nice in that the string could be quantized exactly, at least perturbatively in g_s. In more complicated cases, we won't be so lucky, and we will have to limit ourselves to the less powerful effective field theory methods, potentially missing phenomena such as this non-abelian gauge group enhancement.

1.1.2 Type II superstrings

Supersymmetric world-sheet action

The world-sheet action (1.1.13) can be made supersymmetric. At the most basic level, this means that we promote the $x^M(\sigma)$ to a function of σ and of new formal coordinates $\theta^\pm$ that anticommute: $\theta^+\theta^- = -\theta^-\theta^+$, $(\theta^\pm)^2 = 0$. The Taylor expansion in the new coordinates truncates:

$$X^M = x^M + \theta^+\psi_+^M + \theta^-\psi_-^M + \theta^+\theta^- F^M\,. \tag{1.1.27}$$

We can also introduce the derivative operators

$$D_\pm = \partial_{\theta^\pm} + \mathrm{i}\theta^\pm\partial_\pm\,, \qquad \partial_\pm \equiv \partial_{\sigma^\pm}\,. \tag{1.1.28}$$

Then, (1.1.4), for example, is replaced by

$$S^{1,1}_{\mathrm{F1},g} = -\frac{1}{2}T_{\mathrm{F1}}\int_\Sigma \mathrm{d}^2\sigma\mathrm{d}^2\theta(g+B)_{MN}(X)D_+X^M D_-X^N\,, \tag{1.1.29}$$

with the integration rule $\int \mathrm{d}\theta^\pm\theta^\pm = 1$, $\int \mathrm{d}\theta^\pm 1 = \int \mathrm{d}\theta^\pm\theta^\mp = 0$. We also added the contribution from B. The terms (1.1.10) can also be supersymmetrized in this way. The final result is quite messy for a general background where g_{MN} and B_{MN} are arbitrary; it can be found, for example, in [27, sec. 6.3.1]. For example, it contains a kinetic term

$$g_{MN}(\psi_+^M\partial_-\psi_+^N + \psi_-^M\partial_+\psi_-^N) \tag{1.1.30}$$

for the world-sheet fermions $\psi_\pm^M$.[3] The F^M in (1.1.27) are auxiliary fields: they have no kinetic term, and can be replaced with the solutions of their equations of motion.

Since we have introduced a single θ^+ and a single θ^-, the resulting model is said to have $\mathcal{N} = (1,1)$ supersymmetry. Any two-dimensional bosonic model can be promoted to such a model. In the context of compactifications, one often needs to separate external and internal dimensions, and the supersymmetrization of the world-sheet model in the latter has more supercharges; a common case one needs is $\mathcal{N} = (2,2)$. This is more challenging to achieve, because such *extended* supersymmetry requires that one combine the x^M with each other in pairs. Such a pairing is reminiscent of the idea of complex coordinates, and is at the root of why differential geometry is useful for compactifications. This idea will return in Chapter 9.

[3] The bosonic world-sheet indices $\pm$ are conceptually not the same as the $\pm$ on the fermions, denoting chirality. To emphasize the difference, some authors change the world-sheet indices to $+ \to +\!\!\!+$ and $- \to =$. This also has the benefit that every term in a world-sheet will then have an equal number of pluses and minuses; see for example [28].

Equation (1.1.29) is called the *Neveu–Schwarz–Ramond* (NSR) model. While we introduced it by supersymmetrizing the world-sheet action, we will see later that the resulting spacetime theory also has the much more nontrivial property of *spacetime* supersymmetry.

Spectrum

Even around flat space, the spectrum of (1.1.29) is now more complicated because it depends on what we impose on the fermionic $\psi^M_\pm$. Since a fermion should only get back to itself after a 4π rotation, under 2π we can impose either periodic or antiperiodic boundary conditions, called Neveu–Schwarz (NS) and Ramond (R) respectively. These can be imposed independently on the $\psi^M_\pm$, leading to four sectors: NSNS, NSR, RNS, and RR. The spectrum has to be analyzed in each sector separately, because the Fourier modes for the $\psi^M_\pm$ behave differently in each.

In the NS sector, the fermionic Fourier modes are $b^M_{-i-1/2}$, $i \geq 0$. The two lowest-lying states are

$$|0\rangle_{\mathrm{NS}}\,, \qquad \mathbf{1}\,, \qquad m^2 = \frac{1}{8 l_s^2}(2-d)\,; \tag{1.1.31a}$$

$$b^M_{-1/2}|0\rangle_{\mathrm{NS}}\,, \qquad \mathbf{8}_{\mathrm{V}}\,, \qquad m^2 = \frac{1}{l_s^2}\left(\frac{(2-d)}{8}+1\right)\,. \tag{1.1.31b}$$

We have also indicated what representation these states form under the compact part $\mathrm{SO}(d-2) = \mathrm{SO}(8)$ of the massless little group (1.1.1). For (1.1.31b), the subscript "V" is because there are two more dimension-eight representations of SO(8), which will soon play a role too.

In the R sector, the fermionic Fourier modes are d^M_{-i}, $i \geq 0$. In this case, the vacuum has already $m^2 = 0$, but in fact it is not unique: the modes d^M_0 now don't raise the energy, and they act on the space of vacua. These d^M_0 satisfy a *Clifford algebra* $\{d^M_0, d^N_0\} = 2g^{MN}1$, and as a consequence the space of R vacua transforms as a spinor under spacetime symmetries. In Section 2.1, we will attack Clifford algebras and spinors systematically in every dimension; for now, we only state the main features we need, which are quite similar to the properties of gamma matrices in four dimensions.

- Gamma matrices Γ^M can be defined in every dimension as matrices that satisfy $\{\Gamma^M, \Gamma^N\} = 2g^{MN}1$.
- In $d = 10$ dimensions, they are 32×32 matrices; in $d = 8$, they are 16×16.
- The space of spinors on which the Γ^M act is a representation for the Lorentz group; in $d =$ even, it decomposes in two chiralities, for which we introduce indices α, $\dot\alpha$. Multiplication by a single Γ^M changes chirality, so the nonzero blocks are $\Gamma^M_{\alpha\dot\beta}$ and $\Gamma^M_{\dot\alpha\beta}$.
- In both $d = 10$ with Lorentzian signature and $d = 8$ with Euclidean signature, there is a choice of Γ^M that are all real. (This aspect will be treated more specifically in Sections 2.2.3 and 2.3.)

As a representation of the transverse SO(8) in the little group (1.1.1), the R states then form a reducible representation of dimension 16, which further splits in two