

## Probability

This lively introduction to measure-theoretic probability theory covers laws of large numbers, central limit theorems, random walks, martingales, Markov chains, ergodic theorems, and Brownian motion. Concentrating on the results that are the most useful for applications, this comprehensive treatment is a rigorous, measure theory–based graduate text and reference. Operating under the philosophy that the best way to learn probability is to see it in action, the book contains many extended examples that apply the theory to concrete applications. Readers learn to recognize when a method works and, more important, when it does not.

This fifth edition contains a new chapter on multidimensional Brownian motion and its relationship to PDEs, an advanced topic that is finding new applications. Setting the foundation for this expansion, Chapter 7 now features a proof of Itô's formula. Key exercises that previously were simply proofs left to the reader have been directly inserted into the text as lemmas. The new edition also reinstates discussion about the central limit theorem for martingales and stationary sequences.

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## Contents

<i>Preface</i>	<i>page xi</i>
<b>1 Measure Theory</b>	<b>1</b>
1.1 Probability Spaces	1
1.2 Distributions	8
1.3 Random Variables	13
1.4 Integration	15
1.5 Properties of the Integral	21
1.6 Expected Value	25
1.6.1 Inequalities	25
1.6.2 Integration to the Limit	26
1.6.3 Computing Expected Values	28
1.7 Product Measures, Fubini's Theorem	33
<b>2 Laws of Large Numbers</b>	<b>37</b>
2.1 Independence	37
2.1.1 Sufficient Conditions for Independence	39
2.1.2 Independence, Distribution, and Expectation	41
2.1.3 Sums of Independent Random Variables	43
2.1.4 Constructing Independent Random Variables	45
2.2 Weak Laws of Large Numbers	48
2.2.1 $L^2$ Weak Laws	48
2.2.2 Triangular Arrays	51
2.2.3 Truncation	53
2.3 Borel-Cantelli Lemmas	58
2.4 Strong Law of Large Numbers	65
2.5 Convergence of Random Series*	69
2.5.1 Rates of Convergence	75
2.5.2 Infinite Mean	76
2.6 Renewal Theory*	78
2.7 Large Deviations*	90
<b>3 Central Limit Theorems</b>	<b>98</b>
3.1 The De Moivre-Laplace Theorem	98
3.2 Weak Convergence	100

	3.2.1	Examples	100
	3.2.2	Theory	102
3.3		Characteristic Functions	108
	3.3.1	Definition, Inversion Formula	108
	3.3.2	Weak Convergence	114
	3.3.3	Moments and Derivatives	116
	3.3.4	Polya's Criterion*	119
	3.3.5	The Moment Problem*	121
3.4		Central Limit Theorems	125
	3.4.1	i.i.d. Sequences	125
	3.4.2	Triangular Arrays	128
	3.4.3	Prime Divisors (Erdős-Kac)*	132
	3.4.4	Rates of Convergence (Berry-Esseen)*	136
3.5		Local Limit Theorems*	140
3.6		Poisson Convergence	145
	3.6.1	The Basic Limit Theorem	145
	3.6.2	Two Examples with Dependence	149
3.7		Poisson Processes	151
	3.7.1	Compound Poisson Processes	154
	3.7.2	Thinning	155
	3.7.3	Conditioning	157
3.8		Stable Laws*	159
3.9		Infinitely Divisible Distributions*	168
3.10		Limit Theorems in $\mathbb{R}^d$	171
<b>4</b>		<b>Martingales</b>	178
	4.1	Conditional Expectation	178
		4.1.1 Examples	180
		4.1.2 Properties	182
		4.1.3 Regular Conditional Probabilities*	185
	4.2	Martingales, Almost Sure Convergence	188
	4.3	Examples	194
		4.3.1 Bounded Increments	194
		4.3.2 Polya's Urn Scheme	196
		4.3.3 Radon-Nikodym Derivatives	197
		4.3.4 Branching Processes	200
	4.4	Doob's Inequality, Convergence in $L^p$ , $p > 1$	203
	4.5	Square Integrable Martingales*	208
	4.6	Uniform Integrability, Convergence in $L^1$	211
	4.7	Backwards Martingales	216
	4.8	Optional Stopping Theorems	221
		4.8.1 Applications to Random Walks	223
	4.9	Combinatorics of Simple Random Walk*	227
<b>5</b>		<b>Markov Chains</b>	232
	5.1	Examples	232
	5.2	Construction, Markov Properties	235



*Contents*

ix

	5.3 Recurrence and Transience	243
	5.4 Recurrence of Random Walks Starred Section	248
	5.5 Stationary Measures	259
	5.6 Asymptotic Behavior	268
	5.7 Periodicity, Tail $\sigma$ -Field*	274
	5.8 General State Space*	278
	5.8.1 Recurrence and Transience	281
	5.8.2 Stationary Measures	281
	5.8.3 Convergence Theorem	282
	5.8.4 GI/G/1 Queue	283
<b>6</b>	<b>Ergodic Theorems</b>	<b>286</b>
	6.1 Definitions and Examples	286
	6.2 Birkhoff's Ergodic Theorem	289
	6.3 Recurrence	293
	6.4 A Subadditive Ergodic Theorem	296
	6.5 Applications	300
<b>7</b>	<b>Brownian Motion</b>	<b>305</b>
	7.1 Definition and Construction	305
	7.2 Markov Property, Blumenthal's 0-1 Law	311
	7.3 Stopping Times, Strong Markov Property	316
	7.4 Path Properties	320
	7.4.1 Zeros of Brownian Motion	320
	7.4.2 Hitting Times	321
	7.5 Martingales	325
	7.6 Itô's Formula*	328
<b>8</b>	<b>Applications to Random Walk</b>	<b>336</b>
	8.1 Donsker's Theorem	336
	8.2 CLTs for Martingales	342
	8.3 CLTs for Stationary Sequences	347
	8.3.1 Mixing Properties	351
	8.4 Empirical Distributions, Brownian Bridge	354
	8.5 Laws of the Iterated Logarithm	360
<b>9</b>	<b>Multidimensional Brownian Motion</b>	<b>364</b>
	9.1 Martingales	364
	9.2 Heat Equation	366
	9.3 Inhomogeneous Heat Equation	368
	9.4 Feynman-Kac Formula	370
	9.5 Dirichlet Problem	373
	9.5.1 Exit Distributions	377
	9.6 Green's Functions and Potential Kernels	379
	9.7 Poisson's Equation	382
	9.7.1 Occupation Times	385
	9.8 Schrödinger Equation	387

<b><i>Appendix A</i></b>	<b>Measure Theory Details</b>	394
A.1	Carathéodory's Extension Theorem	394
A.2	Which Sets Are Measurable?	399
A.3	Kolmogorov's Extension Theorem	402
A.4	Radon-Nikodym Theorem	403
A.5	Differentiating under the Integral	407
<i>References</i>		410
<i>Index</i>		415

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## Preface

Some times the lights are shining on me. Other times I can barely see.  
Lately it occurs to me what a long strange trip its been. (Grateful Dead)

In 1989, when the first edition of the book was completed, my sons David and Greg were 3 and 1, and the cover picture showed the Dow Jones at 2650. The last 29 years have brought many changes, but the song remains the same: “The title of the book indicates that as we develop the theory, we will focus our attention on examples. Hoping that the book would be a useful reference for people who apply probability in their work, we have tried to emphasize the results that are important for applications, and illustrated their use with roughly 200 examples. Probability is not a spectator sport, so the book contains almost 450 exercises to challenge the reader and to deepen their understanding.”

The fifth edition has a number of changes:

- The exercises have been moved to the end of the section. The Examples, Theorems, and Lemmas are now numbered in one sequence to make it easier to find things.
- There is a new chapter on multidimensional Brownian motion and its relationship to PDEs. To make this possible, a proof of Itô’s formula has been added to Chapter 7.
- The lengthy Brownian motion chapter has been split into two, with the second one focusing on Donsker’s theorem, etc. The material on the central limit theorem for martingales and stationary sequences deleted from the fourth edition has been reinstated.
- The four sections of the random walk chapter have been relocated. Stopping times have been moved to the martingale chapter; recurrence of random walks and the arcsine laws to the Markov chain chapter; renewal theory has been moved to Chapter 2.
- Some of the exercises that were simply proofs left to the reader have been put into the text as lemmas. There are a few new exercises.

**Typos** The fourth edition contains a list of the people who made corrections to the first three editions. With apologies to those whose contributions I lost track of, this time I need to thank: Richard Arratia, Benson Au, Swee Hong Chan, Conrado Costa, Nate Eldredge, Steve Evans, Jason Farnon, Christina Goldschmidt, Eduardo Hota, Martin Hildebrand, Shlomo Leventhal, Jan Lieke, Kyle MacDonald, Ron Peled, Jonathan Peterson, Erfan Salavati, Byron Schmuland, Timo Seppalainen, Antonio Carlos de Azevedo Sodre, Shouda Wang, and Ruth Williams. I must confess that Christophe Leuridan pointed one out that I have not corrected. Lemma 3.4.19 incorrectly asserts that the distributions in its statement have mean 0, but their means do not exist. The conclusion remains valid, since they are differentiable at 0. A sixth

edition is extremely unlikely, but you can e-mail me about typos and I will post them on my website.

**Family update** As the fourth edition was being completed, David had recently graduated from Ithaca College and Greg was in his last semester at MIT applying to graduate school in computer science. Now, eight years later, Greg has graduated from Berkeley University, and is an assistant professor in the Computer Science department at the University of Texas in Austin. Greg works in the field of machine learning, specifically natural language processing. No, I don't know what that means, but it seems to pay well. David got his degree in journalism. After an extensive job search process and some freelance work, David has settled into a steady job working for a company that produces newsletters for athletic directors and trainers.

In the summer of 2010, Susan and I moved to Durham. Since many people think that the move was about the weather, I will mention that during our first summer it was 104 degrees (and humid!) three days in a row. Yes, it almost never snows here, but when it does, three inches of snow (typically mixed with ice) will shut down the whole town for four days. It took some time for us to adjust to the Durham/Chapel Hill area, which has about 10 times as many people as Ithaca and is criss-crossed by freeways, but we live in a nice quiet neighborhood near the campus. Susan enjoys volunteering at the Sarah P. Duke gardens and listening to their talks about the plants of North Carolina and future plans for the gardens.

As I write this, it is the last week before school starts.