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HIGHER CATEGORIES AND HOMOTOPICAL ALGEBRA

This book provides an introduction to modern homotopy theory through the lens of higher categories after Joyal and Lurie, giving access to methods used at the forefront of research in algebraic topology and algebraic geometry in the twenty-first century. The text starts from scratch – revisiting results from classical homotopy theory such as Serre's long exact sequence, Quillen's theorems A and B, Grothendieck's smooth/proper base change formulas and the construction of the Kan–Quillen model structure on simplicial sets – and develops an alternative to a significant part of Lurie's definitive reference *Higher topos theory*, with new constructions and proofs, in particular, the Yoneda lemma and Kan extensions. The strong emphasis on homotopical algebra provides clear insights into classical constructions such as calculus of fractions, homotopy limits and derived functors, which are revisited in this enhanced context.

For graduate students and researchers from neighbouring fields, this book is a user-friendly guide to the advanced tools that the theory provides for applications in such areas as algebraic geometry, representation theory, algebra and logic.

Denis-Charles Cisinski is Professor of Mathematics at the University of Regensburg, Germany. His research focuses on homotopical algebra, category theory, K-theory and the cohomology of schemes. He is also the author of a monograph entitled *Les préfaisceaux comme modèles des types d'homotopie*.

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Higher Categories and Homotopical Algebra

DENIS-CHARLES CISINSKI
Universität Regensburg, Germany

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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108473200
DOI: 10.1017/9781108588737

First published 2019

Printed and bound in Great Britain by Clays Ltd, Elcograf S.p.A.

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Cisinski, Denis-Charles, author.
Title: Higher categories and homotopical algebra / Denis-Charles Cisinski
(Universitat Regensburg, Germany).
Description: Cambridge ; New York, NY : Cambridge University Press, 2019. |
Series: Cambridge studies in advanced mathematics |
Includes bibliographical references and index.
Identifiers: LCCN 2018048001 | ISBN 9781108473200 (hardback : alk. paper)
Subjects: LCSH: Homotopy theory. | Algebra, Homological. |
Categories (Mathematics) | Presheaves.
Classification: LCC QA612.7 .C5645 2019 | DDC 514/.24–dc23
LC record available at https://lccn.loc.gov/2018048001

ISBN 978-1-108-47320-0 Hardback

CAMBRIDGE
Cambridge University Press
978-1-108-47320-0 — Higher Categories and Homotopical Algebra
Denis-Charles Cisinski
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For Isaac and Noé

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Preface

A Couple of Perspectives and a Tribute

The aim of this book is to introduce the basic aspects of the theory of ∞-categories: a homotopy-theoretic variation on category theory, designed to implement the methods of algebraic topology in broader contexts, such as algebraic geometry [TV05, TV08, Lur09, Lur17] or logic [Uni13, KL16, Kap17]. The theory of ∞-categories is not only a new approach to the foundations of mathematics: it appears in many spectacular advances, such as the proof of Weil's conjecture on Tamagawa numbers over function fields by Lurie and Gaitsgory, or the modern approach to p-adic Hodge theory by Bhatt, Morrow and Scholze, for instance.

For pedagogical reasons, but also for conceptual reasons, a strong emphasis is placed on the following fact: the theory of ∞-categories is a semantic interpretation of the formal language of category theory.[1] This means that one can *systematically* make sense of any statement formulated in the language of category theory in the setting of ∞-categories.[2]

We also would like to emphasise that the presence of homotopical algebra in this book is not as an illustration, nor as a source of technical devices: it is

[1] To be precise, the language of category theory is the one provided by a Cartesian closed category endowed with an involution $X \mapsto X^{\mathrm{op}}$, called the 'opposite category functor', a monoidal structure defined by a 'join operation' $*$, whose unit is the initial object, and which is symmetric up to the opposite operation: $X * Y = (Y^{\mathrm{op}} * X^{\mathrm{op}})^{\mathrm{op}}$. Furthermore, for each object Y, we have the slice functor, obtained as a right adjoint of the functor $X \mapsto (Y \to X * Y)$. Finally, there is a final object Δ^0, and we get simplices by iterating the join operation with it: $\Delta^n = \Delta^0 * \Delta^{n-1}$. Category theory is obtained by requiring properties expressed in this kind of language.

[2] There is, more generally, a theory of (∞, n)-categories: a semantic interpretation of the language of (strict) n-categories (for various ordinals n). The theory of ∞-categories as above is thus the theory of $(\infty, 1)$-categories. Although we shall not say more on these higher versions here, the interested reader might enjoy a look at Baez's lectures [BS10] on these topics.

at the core of basic category theory. In classical category theory, homotopical algebra seems peculiar, because classical homotopy categories do not have (co)limits and are not concrete (i.e. cannot be embedded in the category of sets in a nice way), as the fundamental case of the homotopy category of CW-complexes shows [Fre70]. This is partly why some traditions seem to put classical category theory and classical homotopy theory apart. The story that we want to tell here is that the theory of ∞-categories involves a reunion: with this new semantic interpretation, homotopy theories define ∞-categories with (co)limits, and the classical methods of category theory do apply to them (and the problem of concreteness disappears because ∞-groupoids take the role of sets, not by choice, but under the rule of universal properties). In particular, in this book, model categories will eventually be allowed to be ∞-categories themselves, and we shall observe that the localisation of a model category is also a model category, where the weak equivalences are the invertible maps and the fibrations are all maps (for the reader who might not be familiar with such a language, the present text aims at explaining what such a sentence is about). This means that homotopy theories and their models do live in the same world, which changes dramatically our perspective on them. Finally, one may see homotopical algebra as the study of the compatibility of localisations with (co)limits. And the semantics of ∞-categories makes this a little more savoury because it provides much more powerful and flexible statements. Moreover, the fact that the free completion of a small category by small colimits can be described as the homotopy theory of presheaves of spaces on this category puts homotopical algebra at the very heart of the theory of Kan extensions, and thus of category theory itself. This enlightens many classical results of the heroic days of algebraic topology, such as Eilenberg and Steenrod's characterisation of singular homology, for instance. In some sense, this is the natural outcome of a historical process. Indeed category theory was born as a convenient language to express the constructions of algebraic topology, and the fact that these two fields were separated is a kind of historical accident whose effects only started to fade in the late 1990s, with the rise of ∞-categories as we know them today, after the contributions of André Joyal, Carlos Simpson, Charles Rezk, Bertrand Toën and Gabriele Vezzosi, and of course Jacob Lurie. A pioneer of higher category theory such as Daniel M. Kan was aware of the very fact that category theory should extend to homotopy theory already in the 1950s, and his contributions, all through his mathematical life, through the theory of simplicial categories, with William Dwyer, and, more recently, through the theory of relative categories, with Clark Barwick, for instance, are there to testify to this. The title of this book is less about putting higher category theory and homotopy theory side by side, than observing that higher category theory and homotopical algebra are essentially the same thing. However, a better tribute to Daniel M. Kan might have been to call it *Category theory*, plain and simple.

A Glimpse at the Narrative

As we already wrote above, this text emphasises the fact that the theory of ∞-categories is a semantic interpretation of the language of category theory. But, when it comes to language, there is syntax. And, if category theory is full of identifications which are not strict, such as isomorphisms, equivalences of categories, or even wider notions of weak equivalences, this does not get better with the theory of ∞-categories, which has an even greater homotopy-theoretic flavour. However, the only identification known by syntax is the identity. In practice, this means that we have to introduce various rectification tools, in order to bring back categorical constructions into our favourite language. In Lurie's book [Lur09], which is the standard reference on the subject, by its quality and its scope, this rectification appears early in the text, in several disguises, in the form of Quillen equivalences relating various model structures (e.g. to compare Joyal's model category structure, which encodes the homotopy theory of ∞-categories, with Bergner's model category structure, which expresses Dwyer and Kan's homotopy theory of simplicial categories). These Quillen equivalences consist in introducing several languages together with tools to translate statements from one language to another (for instance, the language provided by the category of simplicial sets, which is used to describe the Joyal model structure, and the language of simplicial categories). This is all good, since we can then extract the most convenient part of each language to express ourselves. But these Quillen equivalences are highly non-trivial: they are complex and non-canonical. And since they introduce new languages, they make unclear which aspects of a statement are independent of the theory we chose to express ourselves.

There are many models to describe ∞-categories, in the same way that there are many ways to describe homotopy types of CW-complexes (such as Kan complexes, or sheaves of sets on the category of smooth manifolds). All these models can be shown to be equivalent. For instance, as already mentioned above, in Lurie's book [Lur09], the equivalence between Kan's simplicial categories and Joyal's quasi-categories is proved and used all through the text, but there are plenty of other possibilities, such as Simpson's Segal categories [Sim12], or Rezk's complete Segal spaces [Rez01]. A reference where one may find all these comparison results is Bergner's monograph [Ber18], to which we should add the beautiful description of ∞-categories in terms of sheaves on an appropriate category of stratified manifolds by Ayala, Francis and Rozenblyum [AFR17]. Riehl and Verity's ongoing series of articles [RV16, RV17a, RV17b] aim at expressing what part of this theory is model independent.

In the present book, we choose to work with Joyal's model category structure only. This means that our basic language is the one of simplicial sets. In fact, the first half of the book consists in following Joyal's journey [Joy08a, Joy08b], step by step: we *literally* interpret the language of category theory in the category of simplicial sets, and observe, with care and wonder, that, although it might look naive at first glance, this defines canonically a homotopy theory such that all the constructions of interest are homotopy invariant in a suitable sense. After some work, it makes perfect sense to speak of the ∞-category of functors between two ∞-categories, to see homotopy types (under the form of Kan complexes) as ∞-groupoids, or to see that fully faithful and essentially surjective functors are exactly equivalences of ∞-categories, for instance. Still in the same vein, one then starts to speak of right fibrations and of left fibrations (i.e. discrete fibrations and discrete op-fibrations, respectively). This is an approach to the theory of presheaves which is interesting by itself, since it involves (generalisations of) Quillen's theorems A and B, revisited with Grothendieck's insights on homotopy Kan extensions (in terms of smooth base change formulas and proper base change formulas). This is where the elementary part ends, in the precise sense that, to go further, some forms of rectification procedure are necessary.

In classical category theory, rectification procedures are most of the time provided by (a variation on) the Yoneda lemma. In Lurie's work as well: the rectification (straightening) of Cartesian fibrations into simplicial contravariant functors is widely used, and this is strongly related to a homotopy-theoretic version of the Yoneda lemma for 2-categories.[3] Rectification is a kind of internalisation: we want to go from ∞-groupoids (or ∞-categories), seen as objects of the theory of ∞-categories, to objects of a suitable '∞-category of ∞-groupoids' (or 'of ∞-categories'). This step is non-trivial, but it is the only way we can see how objects defined up to homotopy are uniquely (and thus coherently) determined in a suitable sense. For instance, externally, the composition of two maps in an ∞-category C, is only well defined up to homotopy (i.e. there is a contractible space of choices) in the sense that, given

[3] There is no need to understand this to go through this book, but for the sake of completeness, let us explain what we mean here. From a Grothendieck fibration $p\colon X \to A$, we can produce a presheaf of categories F on A by defining $F(a)$ as the category of Cartesian functors from the slice category A/a to X (over A) for all a. The fact that p and F determine each other is strongly related to the 2-categorical Yoneda lemma, which identifies $F(a)$ with the category of natural transformation from the presheaf represented by a to F, and to its fibred counterpart: there is a canonical equivalence of categories from $F(a)$ to the fibre X_a of p at a.

three objects x, y and z in C, there is a canonical homotopy equivalence

$$C(x, y) \times C(y, z) \leftarrow C(x, y, z)$$

relating the ∞-groupoid $C(x, y, z)$ of pairs of maps of the form $x \to y \to z$, equipped with a choice of composition $x \to z$, with the product of the ∞-groupoid $C(x, y)$ of maps of the form $x \to y$ with the ∞-groupoid $C(y, z)$ of maps of the form $y \to z$, and there is a tautological composition law

$$C(x, y, z) \to C(x, z)\,.$$

Composing maps in C consists in *choosing* an inverse of the homotopy equivalence above and then applying the tautological composition law. In the case where C is an ordinary category, i.e. when $C(x, y)$ is a set of maps, the composition law is well defined because there really is a unique inverse of a bijective map. The fact that the composition law is well defined and associative in such an ordinary category C implies that the assignment $(x, y) \mapsto C(x, y)$ is actually a functor from $C^{\mathrm{op}} \times C$ to the category of sets. But, when C is a genuine ∞-category, such an assignment is not a functor any more. This is due to the fact that the above is expressed in the language of the category of ∞-categories (as opposed to the ∞-category of ∞-categories), so that the assignment $(x, y) \mapsto C(x, y)$ remains a functional from the set of pairs of objects of C to the collection of ∞-groupoids, seen as objects of the category of ∞-categories. Asking for functoriality is then essentially meaningless. However, internally, such compositions all are perfectly well defined in the sense that there is a genuine Hom functor with values in the ∞-category of ∞-groupoids: there is an appropriately defined ∞-category $\mathcal{S}$ of ∞-groupoids and a functor

$$\mathrm{Hom}_C : C^{\mathrm{op}} \times C \to \mathcal{S}\,.$$

Of course, for the latter construction to be useful, we need to make a precise link between ∞-groupoids, and the objects of $\mathcal{S}$, so that $C(x, y)$ corresponds to $\mathrm{Hom}_C(x, y)$ in a suitable way. And there is no easy way to do this.

Another example: the (homotopy) pull-back of Kan fibrations becomes a strictly associative operation once interpreted as composition with functors with values in the ∞-category of ∞-groupoids. And using the Yoneda lemma (expressed with the functor Hom_C above), this provides coherence results for pull-backs in general. More precisely, given a small ∞-groupoid X with corresponding object in $\mathcal{S}$ denoted by x, there is a canonical equivalence of ∞-groupoids between the ∞-category of functors $\underline{\mathrm{Hom}}(X, \mathcal{S})$ and the slice ∞-category $\mathcal{S}/x$ (this extends the well-known fact that the slice category of sets over a given small set X is equivalent to the category of X-indexed families of sets). Given a functor between small ∞-groupoids $F\colon X \to Y$ corresponding to a map $f\colon x \to y$ in $\mathcal{S}$, the pull-back functor

$$\mathcal{S}/y \to \mathcal{S}/x\,, \quad (t \to y) \mapsto (x \times_y t \to x)$$

corresponds to the functor

$$\underline{\mathrm{Hom}}(Y, \mathcal{S}) \to \underline{\mathrm{Hom}}(X, \mathcal{S})\,, \quad \Phi \mapsto \Phi F\,.$$

The associativity of composition of functors in the very ordinary category of ∞-categories thus explains how the correspondence

$$\underline{\mathrm{Hom}}(X, \mathcal{S}) \simeq \mathcal{S}/x$$

is a way to rectify the associativity of pull-backs of ∞-groupoids which only holds up to a canonical invertible map.

Rectification thus involves a procedure to construct and compute functors with values in the ∞-category of ∞-groupoids, together with the construction of a Hom functor (i.e. of the Yoneda embedding). In this book, we avoid non-trivial straightening/unstraightening correspondences which consist in describing ∞-categories through more rigid models. Instead, we observe that there is a purely syntactic version of this correspondence, quite tautological by nature, which can be interpreted homotopy-theoretically. Indeed, inspired by Voevodsky's construction of a semantic interpretation of homotopy type theory with a univalent universe within the homotopy theory of Kan complexes [KL16], we consider the universal left fibration. The codomain of this universal left fibration, denoted by $\mathcal{S}$, has the property that there is an essentially tautological correspondence between maps $X \to \mathcal{S}$ and left fibrations with small fibres $Y \to X$. In particular, the objects of $\mathcal{S}$ are nothing other than small ∞-groupoids (or, equivalently, small Kan complexes). In the context of ordinary category theory, such a category $\mathcal{S}$ would be the category of sets. In this book, we prove that, as conjectured by Nichols-Barrer [NB07], $\mathcal{S}$ is an ∞-category which is canonically equivalent to the localisation of the category of simplicial sets by the class of weak homotopy equivalences (hence encodes the homotopy theory of CW-complexes). Furthermore, the tautological correspondence alluded to above can be promoted to an equivalence of ∞-categories, functorially in any ∞-category X: an equivalence between an appropriate ∞-category of left fibrations of codomain X and the ∞-category of functors from X to $\mathcal{S}$. Even better, the ∞-category of functors from (the nerve of) a small category I to $\mathcal{S}$ is the localisation of the category of functors from I to simplicial sets by the class of levelwise weak homotopy equivalences. This description of the ∞-category of ∞-groupoids is highly non-trivial, and subsumes the result of Voevodsky alluded to above, about the construction of univalent universes within the homotopy theory of Kan complexes. But it has the advantage that the rectification of left fibrations is done without using the introduction of an extra language, and thus may be used at a rather early stage of the development of the theory of ∞-categories, while keeping an elementary level of expression.

In order to promote the correspondence between left fibrations $Y \to X$ and functors $X \to \mathcal{S}$ to an equivalence of ∞-categories, we need several tools. First, we extend this correspondence to a homotopy-theoretic level: we prove an equivalence of moduli spaces, i.e. we prove that equivalent left fibrations correspond to equivalent functors with values in $\mathcal{S}$ in a coherent way.[4] Subsequently, to reach an equivalence of ∞-categories, we need a series of results which are of interest themselves. We provide an *ad hoc* construction of the Yoneda embedding; this can be done quite explicitly, but the proof that it satisfies the very minimal properties we expect involves non-obvious computations, which we could only explain to ourselves by introducing a bivariant version of left fibrations. Then we develop, in the context of ∞-categories, all of classical category theory (the Yoneda lemma, the theory of adjoint functors, extensions of functors by colimits, the theory of Kan extensions) as well as all of classical homotopical algebra (localisations, calculus of fractions, ∞-categories with weak equivalences and fibrations, Reedy model structures, derived functors, homotopy limits). All these aspects are carried over essentially in the same way as in ordinary category theory (this is what internalisation is good for). The only difference is that inverting weak equivalences in complete ∞-categories gives, under suitable assumptions (e.g. axioms for complete model categories) ∞-categories with small limits. Furthermore, we have the following coherence property: the process of localisation for these commutes with the formation of functor categories (indexed by small 1-categories). This means that in the context of ∞-categories, the notions of homotopy limit and of limit are not only analogous concepts: they do coincide (in particular, homotopy limits, as usually considered in algebraic topology, really are limits in an appropriate ∞-category). Similarly, there are coherence results for finite diagrams. For instance, inverting maps appropriately in ∞-categories with finite limits commutes with the formation of slices. From all this knowledge comes easily the ∞-categorical correspondence between left fibrations $Y \to X$ and functors $X \to \mathcal{S}$. Furthermore, in the case where X is the nerve of a small category A, we observe immediately that the ∞-category of functors $X \to \mathcal{S}$ is the localisation of the category of simplicial presheaves on A by the fibrewise weak homotopy equivalences, which puts classical homotopy theory in perspective within ∞-category theory.

[4] Another way to put it, for type theorists, is that we prove Voevodsky's univalence axiom for the universal left fibration.

A Few Words on the Ways We May Read This Book

Although it presents an alternative approach to the basics of the theory of ∞-categories, and even contains a few new results which might make them of interest to some readers already familiar with higher categories after Joyal and Lurie, this book is really meant to be an introduction to the subject. It is written linearly, that is, following the logical order, which also corresponds to what was actually taught in a two-semester lecture series, at least for most of it. We have aimed at providing complete constructions and proofs, starting from scratch. However, a solid background in algebraic topology or in category theory would certainly help the reader: the few examples only appear at the very end, and, when we introduce a concept, we usually do not give any historical background nor pedestrian justification. We have tried to make clear why such concepts are natural generalisations of siblings from category theory, though. Despite this, apart from a few elementary facts from standard category theory, such as the contents of Leinster's book [Lei14] or parts of Riehl's [Rie17], there are no formal prerequisites for reading this text. A very few technical results, generally with an elementary set-theoretic flavour, are left as exercises, but always with a precise reference where to find a complete proof. In particular, we do not even require any previous knowledge of the classical homotopy theory of simplicial sets, nor of Quillen's model category structures. In fact, even the Kan–Quillen model category structure, corresponding to the homotopy theory of Kan complexes, is constructed in detail, as a warm-up to construct the Joyal model category structure, which corresponds to the homotopy theory of ∞-categories. We also revisit several classical results of algebraic topology, such as Serre's long exact sequence of higher homotopy groups, as well as Quillen's famous theorem A and theorem B. These well-known results are proven in full because they appear in this book in a rather central way. For instance, in order to prove that a functor is an equivalence of ∞-categories if and only if it is fully faithful and essentially surjective, one may observe that the particular case of functors between higher groupoids (i.e. Kan complexes) is a corollary of Serre's long exact sequence. Interestingly enough, the general case follows from this groupoidal version. Similarly, the account we give of Quillen's theorem A is in fact a preparation of the theory of Kan extension, and Quillen's theorem B is a way to understand locally constant functors (which will be a technical but fundamental topic in the computation of localisations).

For the readers who already know the basics of ∞-category theory (e.g. the five first chapters of [Lur09]), parts of Chapters 4, 5 and 6 might still be of interest, since they give an account of the basics which differs from Lurie's treatment. But such readers may go directly to Chapter 7, which deals with the general interpretation of homotopical algebra within the theory of higher categories. The treatment we give of homotopical techniques in this last chapter gives robust and rather optimal tools to implement classical homotopy theories in higher categories. This is a nice example which shows that apparently abstract concepts, such as that of Kan extensions, can be used intrinsically (without apparently more explicit tools, such a homotopy coherent nerves) to organise a theory (e.g. the localisation of higher categories) both conceptually and effectively (i.e. producing computational tools).

One of the interests of using a single formalism which is a literal semantic interpretation of the language of category theory is that, although the proofs can be rather intricate, most of the statements made in this book are easy enough to understand, at least for any reader with some knowledge of category theory. This hopefully should help the reader, whether she or he wants to read only parts of the book, or to follow it step by step. Furthermore, each chapter starts with a detailed description about its purposes and contents. This is aimed at helping the reader to follow the narrative as well as to facilitate the use of the book for reference.

Finally, as all introductions, this book ends when everything begins. The reader is then encouraged to go right away to Lurie's realm. And beyond.

Acknowledgments

This book grew out of several lectures I gave on these topics: a one-semester course at Université Paris 13 in 2009, a mini-course in the Winter School 'Higher Structures in Algebraic Analysis', at the University of Padova in 2014, and a full-year course during the academic year 2016–2017, at the University of Regensburg. While writing this book, I was partially supported by the SFB 1085 'Higher Invariants' funded by the Deutsche Forschungsgemeinschaft (DFG).

I wish to thank particularly Dimitri Ara, Christoph Eibl, Kévin François, Andrea Gagna, Adeel Khan, Han-Ung Kufner, Markus Land, Hoang Kim Nguyen, Marc Nieper-Wißkirchen, Benedikt Preis and George Raptis, for their questions and comments. Many discussions with Uli Bunke certainly provided motivation and inspiration. My knowledge of, and taste for, homotopy theory owe a lot to Georges Maltsiniotis, and this text bears witness of that debt (in particular, my understanding of Grothendieck's views on homotopy theory, especially on derivators and smooth/proper base change formulas, had a great influence on the way I wrote this text). I remember great conversations with Mark Weber and Michael Batanin on cosmoi and Yoneda structures, and these certainly made

their way here. Working with Ieke Moerdijk, a few years ago, on ∞-operads as a generalisation of ∞-categories, was an opportunity to think about the foundations of this theory, and this contributed to my understanding of the subject. Joseph Tapia renewed my interest in logic, in particular, in the dialectic between syntax and semantics. N. Bourbaki gave me a great opportunity to think again about these matters by asking me to give a colloquium talk about Lurie's book on higher topoi. Finally, the influence of many open and friendly discussions with André Joyal cannot be overestimated.