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Prelude

This short chapter is meant to introduce the definition of ∞-categories. However, it starts with a recollection on presheaves of sets on a small category, on the Yoneda lemma, as well as on the ramifications of the latter through extensions of functors by colimits (a particular case of left Kan extensions). This recollection is important because the main language we will use in this book is the one of presheaves of sets, since ∞-categories will be defined as simplicial sets with certain properties, and since simplicial sets are presheaves. On the other hand, extending functors by colimits via presheaves in the setting of ∞-categories may be seen as one of our main goals. In fact, it is probably what underlies the narrative all through this book.

The rest of the chapter recounts the basic features that allow one to understand the cellular structure of simplicial sets, as well as Grothendieck's description of nerves of small categories within simplicial sets. Then come the definitions of ∞-categories and of ∞-groupoids. We see that all Kan complexes are ∞-groupoids (the converse is true but non-trivial and will only be proved in the next chapter), and therefore see that the algebra of paths in topological spaces define ∞-groupoids. The proof of the theorem of Boardmann and Vogt, which describes the category associated to an ∞-category rather explicitly, is quite enlightening, as it is also a first test which strongly indicates that interpreting the language of category theory within the category of simplicial sets is sound.

1.1 Presheaves

Presheaves will reappear in this book many times, and in many disguises. This is the way we express ourselves, at least whenever we use the language of category theory, because of the ubiquitous use of the Yoneda lemma (which will be recalled below). However, the more we go into homotopical algebra, the more we will see this apparently innocent and rather formal looking result, and the more we will see how the Yoneda lemma ramifies into many refinements. We will recall here the basic results needed about presheaves (of sets). These will be used as tools right away, but they also will be revisited with the lenses of homotopical algebra, over and over again. The historical references for this part are D. M. Kan's paper [Kan58] (in which the notion of adjoint functor is introduced for the first time), as well as Grothendieck's [SGA72, Exposé I] (the presentation we give here is rather close to the latter).

We write *Set* for the category of sets.

Definition 1.1.1 Let A be a category. A *presheaf* over A is a functor of the form

$$X\colon A^{\mathrm{op}} \to \mathit{Set}.$$

For an object a of A, we will denote by

$$X_a = X(a)$$

the evaluation of X at a. The set X_a will sometimes be called the *fibre* of the presheaf X at a, and the elements of X_a thus deserve the name of *sections* of X over a. For a morphism $u\colon a \to b$ in A, the induced map from X_b to X_a often will be written

$$u^* = X(u)\colon X_b \to X_a\,.$$

If X and Y are two presheaves over A, a *morphism* of presheaves $f\colon X \to Y$ simply is a natural transformation from X to Y. In other words, such a morphism f is determined by a collection of maps $f_a\colon X_a \to Y_a$, such that, for any morphism $u\colon a \to b$ in A, the following square commutes.

$$\begin{array}{ccc} X_a & \xrightarrow{f_a} & Y_a \\ {\scriptstyle u^*}\uparrow & & \uparrow{\scriptstyle u^*} \\ X_b & \xrightarrow{f_b} & Y_b \end{array} \qquad f_a\, u^* = u^*\, f_b\,.$$

Presheaves naturally form a category. This category will be written $\widehat{A}$.

Remark 1.1.2 One checks that a morphism of presheaves $f\colon X \to Y$ is an isomorphism (a monomorphism, an epimorphism) if and only if, for any object a of A, the induced map $f_a\colon X_a \to Y_a$ is bijective (injective, surjective, respectively). Moreover, the evaluation functors $X \mapsto X_a$ preserve both limits and colimits (exercise: deduce this latter property by exhibiting a left adjoint and a right adjoint). As a consequence, if $F\colon I \to \widehat{A}$ is a diagram of presheaves and if X is a presheaf, the property that a cone from X to F (a cocone from

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F to X) exhibits X as a limit (colimit) of F is local in the sense that it can be tested fibrewise. In other words, X is a limit (a colimit) of F if and only if, for any object a of A, the set X_a is a limit (a colimit) of the induced diagram $F_a : I \to Set$, respectively.

Definition 1.1.3 The *Yoneda embedding* is the functor

$$h\colon A \to \widehat{A} \tag{1.1.3.1}$$

whose value at an object a of A is the presheaf

$$h_a = \mathrm{Hom}_A(-, a)\,. \tag{1.1.3.2}$$

In other words, the evaluation of the presheaf h_a at an object c of A is the set of maps from c to a.

Theorem 1.1.4 (Yoneda lemma) *For any presheaf X over A, there is a natural bijection of the form*

$$\mathrm{Hom}_{\widehat{A}}(h_a, X) \xrightarrow{\sim} X_a$$
$$(h_a \xrightarrow{u} X) \mapsto u_a(1_a).$$

Proof We only define the map in the other direction. Given a section s of X over a, we define a collection of morphisms

$$f_c\colon\ \mathrm{Hom}_A(c, a) \to X_c$$

(indexed by objects of A) as follows: for each morphism $u\colon c \to a$, the section $f_c(u)$ is the element $f_c(u) = u^*(s)$. One then checks that this collection defines a morphism $f\colon h_a \to X$, and that the assignment $s \mapsto f$ is a two-sided inverse of the Yoneda embedding. □

Corollary 1.1.5 *The Yoneda embedding $h\colon A \to \widehat{A}$ is a fully faithful functor.*

Notation 1.1.6 The author of this book prefers to write the isomorphism of the Yoneda embedding as an equality; we will often make an abuse of notation by writing again $f\colon a \to X$ for the morphism of presheaves associated to a section $f \in X_a$ (via the Yoneda lemma).

Definition 1.1.7 Let X be a presheaf on a category A. The *category of elements* of X (we also call it the *Grothendieck construction* of X) is the category whose objects are couples (a, s), where a is an object of A, while s is a section of X over a, and whose morphisms $u\colon (a, s) \to (b, t)$ are morphisms $u\colon a \to b$ in A, such that $u^*(t) = s$. If we adopt the abuse of notation of paragraph 1.1.6, this latter condition corresponds, through the Yoneda lemma, to the commutativity of the triangle below.

$$\begin{array}{ccc} h_a & \xrightarrow{\ u\ } & h_b \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle t} & \\ & X & \end{array}$$

The *category of elements* of X is denoted by A/X. It comes equipped with a faithful functor

(1.1.7.1) $$\varphi_X : A/X \to \widehat{A}$$

defined on objects by $\varphi_X(a, s) = h_a$, and on morphisms, by $\varphi_X(u) = u$. There is an obvious cocone from φ_X to X defined by the following collection of maps:

(1.1.7.2) $$s : h_a \to X\,, \quad (a, s) \in \mathrm{Ob}(A/X)\,.$$

A variation on the Yoneda lemma is the next statement.

Proposition 1.1.8 *The collection of maps* (1.1.7.2) *exhibits the presheaf X as the colimit of the functor* (1.1.7.1).

Proof Let Y be another presheaf on the category A. We have to show that the operation of composing maps from X to Y with the maps (1.1.7.2) defines a (natural) bijection between morphisms from X to Y and cocones from the functor φ_X to Y in the category of presheaves over A. By virtue of the Yoneda lemma, a cocone from φ_X to Y can be seen as a collection of sections

$$f_s \in Y_a\,, \quad (a, s) \in \mathrm{Ob}(A/X)$$

such that, for any morphism $u : (a, s) \to (b, t)$ in A/X, we have the relation $u^*(f_t) = f_s$. This precisely means that the collection of maps

$$\begin{aligned} X_a &\to Y_a\,, \quad a \in \mathrm{Ob}(A) \\ s &\mapsto f_s \end{aligned}$$

is a morphism of presheaves. One then checks that this operation is a two-sided inverse of the operation of composition with the family (1.1.7.2). □

Remark 1.1.9 Until this very moment, we did not mention size (smallness) problems. Well, this is because there were not many. We will come back to size issues little by little. But, whenever we start to be careful with smallness, it is hard to stop. First, when we defined the Yoneda embedding (1.1.3.1), a first problem arose: for this construction to make sense, we need to work with *locally small categories*.[1] We might say: well, maybe we did not formulate things properly, since, for instance, even if their formulations seem to need the property that the category A is locally small, the proofs of the Yoneda lemma (1.1.4) and of its avatar (1.1.8) obviously are valid for possibly large categories. Or we could say: let us restrict ourselves to locally small categories, since, after all, most authors actually require the property of local smallness in the very definition of a category. But Definition 1.1.1 actually provides examples of categories which are not locally small: for a general locally small category A, the category of presheaves over A may not be locally small (exercise: find many examples). And there are other (less trivial but at least as fundamental) categorical constructions which do not preserve the property of being locally small (e.g. localisation). All this means that it might be wiser not to require that all categories are locally small, but, instead, to understand how and why, under appropriate assumptions, certain categorical constructions preserve properties of smallness, or of being locally small. For instance, we can see that, if ever the category A is small,[2] the category of presheaves $\widehat{A}$ is locally small. Moreover, the preceding theorem has the following consequence.

Theorem 1.1.10 (Kan) *Let A be a small category, together with a locally small category $\mathcal{C}$ which has small colimits. For any functor $u\colon A \to \mathcal{C}$, the functor of evaluation at u*

(1.1.10.1) $$u^*: \mathcal{C} \to \widehat{A}\,, \quad Y \mapsto u^*(Y) = (a \mapsto \mathrm{Hom}_{\mathcal{C}}(u(a), Y))$$

has a left adjoint

(1.1.10.2) $$u_!: \widehat{A} \to \mathcal{C}\,.$$

Moreover, there is a unique natural isomorphism

(1.1.10.3) $$u(a) \simeq u_!(h_a)\,, \quad a \in \mathrm{Ob}(A)\,,$$

such that, for any object Y of $\mathcal{C}$, the induced bijection

$$\mathrm{Hom}_{\mathcal{C}}(u_!(h_a), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(u(a), Y)$$

[1] A category is locally small if, for any ordered pair of its objects a and b, morphisms from a to b do form a small set (depending on the set-theoretic foundations the reader would prefer, a small set must either be a set, as opposed to a proper class, or a set which is (in bijection with) an element of a fixed Grothendieck universe). Until we mention universes explicitly (which will happen in the second half of the book), we can be agnostic, at least as far as set theory is concerned. We refer to [Shu08] for an excellent account on the possible set-theoretic frameworks for category theory.

[2] We remind the reader that this means that it is locally small and that its objects also form a small set.

is the inverse of the composition of the Yoneda bijection

$$\mathrm{Hom}_{\mathcal{C}}(u(a), Y) = u^*(Y)_a \simeq \mathrm{Hom}_{\widehat{A}}(h_a, u^*(Y))$$

with the adjunction formula

$$\mathrm{Hom}_{\widehat{A}}(h_a, u^*(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(u_!(h_a), Y)\,.$$

Proof We shall prove that the functor u^* has a left adjoint (the second part of the statement is a direct consequence of the Yoneda lemma). For each presheaf X over A, we choose a colimit of the functor

$$A/X \to \mathcal{C}\,, \quad (a, s) \mapsto u(a)\,,$$

which we denote by $u_!(X)$. When $X = h_a$ for some object a of A, we have a canonical isomorphism $u(a) \simeq u_!(h_a)$ since $(a, 1_a)$ is a final object of A/h_a. Therefore, for any presheaf X over A, and any object Y of $\mathcal{C}$, we have the following identifications:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(u_!(X), Y) &\simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim_{(a,s)} u(a), Y) \\
&\simeq \varprojlim_{(a,s)} \mathrm{Hom}_{\mathcal{C}}(u(a), Y) \\
&\simeq \varprojlim_{(a,s)} \mathrm{Hom}_{\widehat{A}}(h_a, u^*(Y)) \quad \text{by the Yoneda lemma} \\
&\simeq \mathrm{Hom}_{\widehat{A}}(\varinjlim_{(a,s)} h_a, u^*(Y)) \\
&\simeq \mathrm{Hom}_{\widehat{A}}(X, u^*(Y)) \quad \text{by Proposition 1.1.8.}
\end{aligned}$$

In other words, the object $u_!(X)$ (co)represents the functor $\mathrm{Hom}_{\widehat{A}}(X, u^*(-))$. □

Remark 1.1.11 The functor $u_!$ will be called the *extension of u by colimits*. In fact, any colimit preserving functor $F : \widehat{A} \to \mathcal{C}$ is isomorphic to a functor of the form $u_!$ as above. More precisely, for any such colimit preserving functor F, if we put $u(a) = F(h_a)$, there is a unique natural isomorphism $u_!(X) = F(X)$ which is the identity whenever the presheaf X is representable (exercise). For instance, for $\mathcal{C} = \widehat{A}$, the identity of $\widehat{A}$ is (canonically isomorphic to) $h_!$, for h the Yoneda embedding.

Corollary 1.1.12 *Any colimit preserving functor* $\widehat{A} \to \mathcal{C}$ *has a right adjoint.*

Proof It is sufficient to consider functors of the form $u_!$, for a suitable functor $u : A \to \mathcal{C}$ (see the preceding remark). Therefore, by virtue of Theorem 1.1.10, it has a right adjoint, namely u^*. □

Notation 1.1.13 Let A be a small category. Then the category of presheaves over A is Cartesian closed: for any presheaves X and Y, there is an internal Hom, that is a presheaf $\underline{\mathrm{Hom}}(X,Y)$ together with natural bijections

$$\mathrm{Hom}_{\widehat{A}}(T, \underline{\mathrm{Hom}}(X,Y)) \simeq \mathrm{Hom}_{\widehat{A}}(T \times X, Y)\,.$$

As can be seen from Theorem 1.1.10 and Remark 1.1.11, this object is defined by the formula

$$\underline{\mathrm{Hom}}(X,Y)_a = \mathrm{Hom}_{\widehat{A}}(h_a \times X, Y)\,.$$

Remark 1.1.14 Given a presheaf X, it is equivalent to study maps of codomain X or to study presheaves on the category A/X. To be more precise, one checks that the extension by colimit of the composed functor $A/X \to A \xrightarrow{h} \widehat{A}$ sends the final object of $\widehat{A/X}$ to the presheaf X, and the induced functor

(1.1.14.1) $$\widehat{A/X} \xrightarrow{\sim} \widehat{A}/X$$

is an equivalence of categories. For this reason, even though we will mainly focus on presheaves on a particular category (simplicial sets), it will be convenient to axiomatise our constructions in order to apply them to various categories of presheaves. Equivalence (1.1.14.1) will be at the heart of the construction of the ∞-category of small ∞-groupoids: this will appear in Section 5.2 below, and will be implicitly at the heart of much reasoning all through the second half of this book.

1.2 The Category of Simplicial Sets

We shall write $\mathbf{\Delta}$ for the category whose objects are the finite sets

$$[n] = \{i \in \mathbf{Z} \mid 0 \leq i \leq n\} = \{0, \dots, n\}\,, \quad n \geq 0\,,$$

endowed with their natural order, and whose maps are the (non-strictly) order-preserving maps.

Definition 1.2.1 A *simplicial set* is a presheaf over the category $\mathbf{\Delta}$. We shall write $sSet = \widehat{\mathbf{\Delta}}$ for the category of simplicial sets.

Notation 1.2.2 For $n \geq 0$, we denote by $\Delta^n = h_{[n]}$ the *standard n-simplex* (i.e. the presheaf on $\mathbf{\Delta}$ represented by $[n]$).

For a simplicial set X and an integer $n \geq 0$, we write

(1.2.2.1) $$X_n = X([n]) \simeq \mathrm{Hom}_{sSet}(\Delta^n, X)$$

for the set of *n-simplices of X*. A *simplex* of X is an element of X_n for some non-negative integer n. In agreement with the abuse of notation introduced in paragraph 1.1.6, an n-simplex x of X can also be seen as a morphism of simplicial sets $x\colon \Delta^n \to X$.

For integers $n \geq 1$ and $0 \leq i \leq n$, we let

$$\partial^n_i\colon \Delta^{n-1} \to \Delta^n \tag{1.2.2.2}$$

be the map corresponding to the unique strictly order preserving map from $[n-1]$ to $[n]$ which does not take the value i.

For integers $n \geq 0$ and $0 \leq i \leq n$, the map

$$\sigma^n_i\colon \Delta^{n+1} \to \Delta^n \tag{1.2.2.3}$$

corresponds to the unique surjective map from $[n+1]$ to $[n]$ which takes the value i twice.

Proposition 1.2.3 *The following identities hold:*

$$\partial^{n+1}_j \partial^n_i = \partial^{n+1}_i \partial^n_{j-1} \qquad i < j, \tag{1.2.3.1}$$

$$\sigma^n_j \sigma^{n+1}_i = \sigma^n_i \sigma^{n+1}_{j+1} \qquad i \leq j, \tag{1.2.3.2}$$

$$\sigma^{n-1}_j \partial^n_i = \begin{cases} \partial^{n-1}_i \sigma^{n-2}_{j-1} & i < j, \\ 1_{\Delta^{n-1}} & i \in \{j, j+1\}, \\ \partial^{n-1}_{i-1} \sigma^{n-2}_j & i > j+1. \end{cases} \tag{1.2.3.3}$$

The proof is straightforward.

Remark 1.2.4 One can prove that the category $\mathbf{\Delta}$ is completely determined by the relations above: more precisely, it is isomorphic to the quotient by these relations of the free category generated by the oriented graph which consists of the collection of maps ∂^n_i and σ^n_i (with the $[n]$ as vertices). In other words, a simplicial set can be described as a collection of sets X_n, $n \geq 0$, together with face operators $d^i_n = (\partial^n_i)^*\colon X_n \to X_{n-1}$ for $n \geq 1$, and degeneracy operators $s^i_n = (\sigma^n_i)^*\colon X_n \to X_{n+1}$ satisfying the dual version of the identities above. This pedestrian point of view is often the one taken in historical references.

Notation 1.2.5 For a simplicial set X, we shall write

$$d^i_n = (\partial^n_i)^*\colon X_n \to X_{n-1} \quad \text{and} \quad s^i_n = (\sigma^n_i)^*\colon X_n \to X_{n+1}$$

for the maps induced by the operators ∂^n_i and σ^n_i, respectively.

Although it follows right away from the notion of image of a map of sets, the following property is the source of many good combinatorial behaviours of the category $\mathbf{\Delta}$.

Proposition 1.2.6 *Any morphism* $f\colon \Delta^m \to \Delta^n$ *in* $\mathbf{\Delta}$ *admits a unique factorisation* $f = i\pi$*, into a split epimorphism* $\pi\colon \Delta^m \to \Delta^p$ *followed by a monomorphism (i.e. a strictly order preserving map)* $i\colon \Delta^p \to \Delta^n$*.*

Example 1.2.7 A good supply of simplicial sets comes from the category *Top* of topological spaces (with continuous maps as morphisms). For this, one defines, for each non-negative integer $n \geq 0$, the topological simplex

$$|\Delta^n| = \left\{ (x_1, \dots, x_n) \in \mathbf{R}^n_{\geq 0} \;\middle|\; \sum_{i=1}^{n} x_i \leq 1 \right\} . \tag{1.2.7.1}$$

Given a morphism $f\colon [m] \to [n]$ in $\mathbf{\Delta}$, we get an associated continuous (because affine) map

$$|f|\colon |\Delta^m| \to |\Delta^n|$$

defined by

$$|f|(x_0, \dots, x_m) = (y_0, \dots, y_n), \quad \text{where } y_j = \sum_{i \in f^{-1}(j)} x_i .$$

This defines a functor from Δ to *Top*. Therefore, by virtue of Theorem 1.1.10, we have the *singular complex functor*

$$Top \to sSet\,, \quad Y \mapsto Sing(Y) = ([n] \mapsto \mathrm{Hom}_{Top}(|\Delta^n|, Y)) \tag{1.2.7.2}$$

and its left adjoint, the *realisation functor*

$$sSet \to Top\,, \quad X \mapsto |X|\,. \tag{1.2.7.3}$$

This example already gives an indication of the possible semantics we can apply to simplicial sets. For instance, a 0-simplex $x\colon \Delta^0 \to X$ can be interpreted as a point of X, and a 1-simplex $f\colon \Delta^1 \to X$ as a path in X, from the point $x = d^1_1(f)$ to the point $y = d^0_1(f)$. This is already good, but we shall take into account that the orientation of paths can be remembered. And doing so literally, this will give semantics, in the category of simplicial sets, of the very language of category theory.

1.3 Cellular Filtrations

In this chapter, we shall review the combinatorial properties of simplicial sets which will be used many times to reduce general statements to the manipulation of finitely many operations on standard simplices. However, we shall present an axiomatised version (mainly to deal with simplicial sets over a given simplicial set X, or with bisimplicial sets, for instance). A standard source on this, in the case of simplicial sets themselves, is the appropriate chapter in the book of Gabriel and Zisman [GZ67]. What follows is to axiomatise the constructions and proofs of therein. For a nice axiomatic treatment of this kind of property, an excellent reference is Bergner and Rezk's paper [BR13].

Definition 1.3.1 An *Eilenberg–Zilber category* is a quadruple (A, A_+, A_-, d), where A is a small category, while A_+ and A_- are subcategories of A, and $d\colon \mathrm{Ob}(A) \to \mathbf{N}$ is a function with values in the set of non-negative integers, such that the following properties are verified:

EZ0. Any isomorphism of A is in both A_+ and A_-. Moreover, for any isomorphic objects a and b in A, we have $d(a) = d(b)$.

EZ1. If $a \to a'$ is a morphism in A_+ (in A_-) that is not an identity, then we have $d(a) < d(a')$ (we have $d(a) > d(a')$, respectively).

EZ2. Any morphism $u\colon a \to b$ in A has a unique factorisation of the form $u = ip$, with $p\colon a \to c$ in A_- and $i\colon c \to b$ in A_+.

EZ3. If a morphism $\pi\colon a \to b$ belongs to A_- there exists a morphism $\sigma\colon b \to a$ in A such that $\pi\sigma = 1_b$. Moreover, for any two morphisms in A_- of the form $\pi, \pi'\colon a \to b$, if π and π' have the same sets of sections, then they are equal.

We shall say that an object a of A is of *dimension* n if $d(a) = n$.

Example 1.3.2 The category $\mathbf{\Delta}$ is an Eilenberg–Zilber category, with $\mathbf{\Delta}_+$ the subcategory of monomorphisms, and $\mathbf{\Delta}_-$ the subcategory of epimorphisms, and $d(\Delta^n) = n$.

Example 1.3.3 If A is an Eilenberg–Zilber category, then, for any presheaf X, the category A/X is an Eilenberg–Zilber category: one defines the subcategory $(A/X)_+$ (the subcategory $(A/X)_-$) as the subcategory of maps whose image in A belongs to A_+ (to A_-, respectively), and one puts $d(a, s) = d(a)$.

Example 1.3.4 If A and B are two Eilenberg–Zilber categories, their product is one as well: one defines $(A \times B)_\varepsilon = A_\varepsilon \times B_\varepsilon$ for $\varepsilon \in \{+, -\}$, and one puts $d(a, b) = d(a) + d(b)$.

Let us fix an Eilenberg–Zilber category A.

Definition 1.3.5 Let X be a presheaf over A. A section x of X over some object a of A is *degenerate*, if there exists a map $\sigma\colon a \to b$ in A, with $d(b) < d(a)$, and a section y of X over b, such that $\sigma^*(y) = x$. Such a couple will be called a *decomposition of* x. A section of X is *non-degenerate* if it is not degenerate.

For any integer $n \geq 0$, we denote by $Sk_n(X)$ the maximal subpresheaf of X with the property that, for any integer $m > n$, any section of $Sk_n(X)$ over