1 Mathematical Models in Economics

1.1 Introduction

In this book we use the language of mathematics to describe situations which occur in economics. The motivation for doing this is that mathematical arguments are logical and exact, and they enable us to work out in precise detail the consequences of economic hypotheses. For this reason, mathematical modelling has become an indispensable tool in economics, finance, business and management. It is not always simple to use mathematics, but its language and its techniques enable us to frame and solve problems that cannot be attacked effectively in other ways. Furthermore, mathematics leads not only to numerical (or quantitative) results but, as we shall see, to qualitative results as well.

1.2 A Model of the Market

One of the simplest and most useful models is the description of supply and demand in the market for a single good. This model is concerned with the relationships between two things: the price per unit of the good (usually denoted by $p$) and the quantity of it on the market (usually denoted by $q$). The ‘mathematical model’ of the situation is based on the simple idea of representing a pair of numbers as a point in a diagram, by means of coordinates with respect to a pair of axes. In economics it is customary to take the horizontal axis as the $q$-axis, and the vertical axis as the $p$-axis. Thus, for example, the point with coordinates $(2000, 7)$ represents the situation where 2000 units are available at a price of $7 per unit.

How do we describe demand in such a diagram? The idea is to look at those pairs $(q, p)$ which are related in the following way: if $p$ were the selling price, $q$ would be the demand, that is the quantity which would be sold to consumers at that price. If we fill in on a diagram all the pairs $(q, p)$ related in this way, we get something like Figure 1.1.

We shall refer to this as the demand set $D$ for the particular good. In economics you will learn reasons why it ought to look rather like it does in our diagram, a smooth, downward-sloping curve.

Suppose the demand set $D$ contains the point $(30, 5)$. This means that when the price $p = 5$ is given, then the corresponding demand will be for $q = 30$ units. In general,
Figure 1.1 The demand set.

provided \( D \) has the ‘right shape’, as in Figure 1.1, then for each value of \( p \) there will be a uniquely determined value of \( q \). In this situation we say that \( D \) determines a demand function, \( q^D \). The value written \( q^D(p) \) is the quantity which would be sold if the price were \( p \), so that \( q^D(5) = 30 \), for example.

**Example** Suppose the demand set \( D \) consists of the points \((q, p)\) on the straight line \( 6q + 8p = 125 \). Then for a given value of \( p \) we can determine the corresponding \( q \); we simply rearrange the equation of the line in the form \( q = (125 - 8p)/6 \). So here the demand function is

\[
q^D(p) = \frac{125 - 8p}{6}.
\]

For any given value of \( p \) we find the corresponding \( q \) by substituting in this formula. For example, if \( p = 4 \) we get

\[ q = q^D(4) = \frac{125 - 8 \times 4}{6} = \frac{93}{6} \]

as the quantity.

There is another way of looking at the relationship between \( q \) and \( p \). If we suppose that the quantity \( q \) is given, then the value of \( p \) for which \((q, p)\) is in the demand set \( D \) is the price that consumers would be prepared to pay if \( q \) is the quantity available. From this viewpoint we are expressing \( p \) in terms of \( q \), instead of the other way round. We write \( p^D(q) \) for the value of \( p \) corresponding to a given \( q \), and we call \( p^D \) the inverse demand function.

**Example** Taking the same set \( D \) as before, we can now rearrange the equation of the line in the form \( p = (125 - 6q)/8 \). So

\[
p^D(q) = \frac{125 - 6q}{8}
\]

is the inverse demand function.
Next we turn to the supply side. We assume that there is a supply set $S$ consisting of those pairs $(q, p)$ for which $q$ would be the amount supplied to the market if the price were $p$. There are good economic reasons for supposing that $S$ has the general form shown in Figure 1.2.

If we know the supply set $S$ we can construct the supply function $q^S$ and the inverse supply function $p^S$ in the same way as we did for the demand function and its inverse. For example, if $S$ is the set of points on the line $2q - 5p = -12$, then solving the equation for $q$ and for $p$ we get

$$q^S(p) = \frac{5p - 12}{2} \quad \text{and} \quad p^S(q) = \frac{2q + 12}{5}.$$

### 1.3 Market Equilibrium

The usefulness of a mathematical model lies in the fact that we can use mathematical techniques to obtain information about it. In the case of supply and demand, the most important problem is the following. Suppose we know all about the factors affecting supply and demand in the market for a particular good; in other words, the sets $S$ and $D$ are given. What values of $q$ and $p$ will actually be achieved in the market?

Figure 1.3 makes it clear that the solution is to find the intersection of $D$ and $S$, because that is where the quantity supplied is exactly balanced by the quantity required.

The mathematical symbol for the intersection of the sets $S$ and $D$ is $S \cap D$, and economists refer to $E = S \cap D$ as the equilibrium set for the given market.

Fortunately, there is a simple mathematical technique for finding the equilibrium set; it is the method for solving ‘simultaneous equations’.

**Example** Suppose the sets $D$ and $S$ are, respectively, the sets of pairs $(q, p)$ such that

$$q + 5p = 40 \quad \text{and} \quad 2q - 15p = -20.$$
Then a point \((q^*, p^*)\) which is in the equilibrium set \(E = S \cap D\) must, by definition, be in both \(S\) and \(D\). Thus \((q^*, p^*)\) satisfies the two equations

\[
q^* + 5p^* = 40 \quad \text{and} \quad 2q^* - 15p^* = -20.
\]

The standard technique for solving these equations is to multiply the first one by 2 and subtract it from the second one. This tells us that

\[
2(q^* + 5p^*) - (2q^* - 15p^*) = 2(40) - (-20)
\]

which, on simplification, means \(25p^* = 100\), so that \(p^* = 4\). To find \(q^*\), we can note that \(q^* + 5p^* = 40\) now implies \(q^* = 40 - 5p^* = 40 - 5(4) = 20\). In other words, the equilibrium set \(E\) contains the single point \((20, 4)\).

It is worth remarking that in this example we get a single point of equilibrium because we took the sets \(D\) and \(S\) to be straight lines. It is possible to imagine more complex situations, such as that we shall describe in Example 2.5, where the equilibrium set contains several points, or no points at all.

### 1.4 Excise Tax

Using only the simple techniques developed so far we can obtain some interesting insights into problems in economics. In this section we study the problem of excise tax. Suppose that a government wishes to discourage its citizens from drinking too much whisky. One way to do this is to impose a fixed tax on each bottle of whisky sold. For example, the government may decide that for each bottle of whisky the suppliers sell, they must pay the government \$1. Note that the tax on each unit of the taxed good is a fixed amount, not a percentage of the selling price.

Some very simple mathematics tells us how the selling price changes when an excise tax is imposed.
Example In the previous example the demand and supply functions are given by

\[ q^D(p) = 40 - 5p \quad \text{and} \quad q^S(p) = \frac{15}{2}p - 10, \]

and the equilibrium price is \( p^* = 4 \). Suppose that the government imposes an excise tax of \( T \) per unit. How does this affect the equilibrium price?

The answer is found by noting that, if the new selling price is \( p \), then, from the supplier’s viewpoint, it is as if the price were \( p - T \), because the supplier’s revenue per unit is not \( p \), but \( p - T \). In other words, the supply function has changed: when the tax is \( T \) per unit, the new supply function \( q^{S_T} \) is given by

\[ q^{S_T}(p) = q^S(p - T) = \frac{15}{2}(p - T) - 10. \]

Of course the demand function remains the same. The new equilibrium values \( q^T \) and \( p^T \) satisfy the equations

\[ q^T = 40 - 5p^T \quad \text{and} \quad q^T = q^{S_T}(p^T) = \frac{15}{2}(p^T - T) - 10. \]

Eliminating \( q^T \) we get

\[ 40 - 5p^T = \frac{15}{2}(p^T - T) - 10. \]

Rearranging this equation, we obtain

\[ \left(5 + \frac{15}{2}\right)p^T = 50 + \frac{15}{2}T, \]

and so we have a new equilibrium price of

\[ p^T = 4 + \frac{3}{5}T. \]

The corresponding new equilibrium quantity is

\[ q^T = 40 - 5p^T = 20 - 3T. \]

For example, if \( T = 1 \), the equilibrium price rises from 4 to 4.6 and the equilibrium quantity falls from 20 to 17. Unsurprisingly, the selling price has risen and the quantity sold has fallen. But note that, although the tax is \( T \) per unit, the selling price has risen not by the full amount \( T \), but by the fraction \( 3/5 \) of \( T \). In other words, not all of the tax is passed on to the consumer.

1.5 Comments

1. Economics tells us why the supply and demand sets ought to have certain properties. Mathematics tells us what we can deduce from those properties and how to do the calculations.
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2. Mathematics also enables us to develop additional features of the model. In the case of supply and demand, we might ask questions such as the following:
   - What happens if conditions change, so that the supply and demand sets are altered slightly?
   - If the equilibrium is disturbed for some reason, what is the result?
   - How do the suppliers and consumers arrive at the equilibrium?

A typical instance of the first question is the excise tax discussed above. In this book we shall develop the mathematical techniques needed to deal with many other instances of these questions.

Worked Examples

Example 1.1 If \( x - 2y = 3 \) and \( 3x + 5y = 20 \), what are \( x \) and \( y \)?

Solution

We eliminate \( y \) from the simultaneous equations. (An alternative, and equally valid, first step would be to eliminate \( x \).) To do this, we multiply the first equation by 5 and the second by 2, obtaining the two equations

\[
\begin{align*}
5x - 10y &= 15, \\
6x + 10y &= 40.
\end{align*}
\]

Adding these, we have

\[
(5x - 10y) + (6x + 10y) = 15 + 40, \quad 11x = 55,
\]

so that \( x = 5 \). Given this, we can use the first equation, \( x - 2y = 3 \), to determine \( y \):

\[
y = (x - 3)/2 = (5 - 3)/2 = 1.\]

Therefore, \( x = 5 \) and \( y = 1 \).

Example 1.2 Suppose that the supply and demand sets, \( S \) and \( D \), for a particular market are described as follows: \( S \) consists of the pairs \( (q, p) \) such that \( 2p - 3q = 12 \) and \( D \) consists of the pairs \( (q, p) \) such that \( 2p + q = 20 \). Determine the supply function \( q^S(p) \), the inverse supply function \( p^S(q) \), the demand function \( q^D(p) \) and the inverse demand function \( p^D(q) \). Sketch \( S \) and \( D \) and determine the equilibrium set \( E = S \cap D \). Comment briefly on the interpretation of the results.

Solution

The supply function is obtained by expressing quantity in terms of price for points in the supply set. We have \( 2p - 3q = 12 \), and rearranging this gives \( q = \frac{2}{3}p - 4 \). Thus

\[
q^S(p) = \frac{2}{3}p - 4.
\]

Similarly,

\[
q^D(p) = 20 - 2p.
\]
Worked Examples

Figure 1.4 The demand set and the supply set for Example 1.2.

To obtain the inverse supply and inverse demand functions, we express \( p \) in terms of \( q \) on the supply and demand sets. Thus, the inverse supply function is

\[
p^S(q) = 6 + \frac{3}{2}q
\]

and the inverse demand function is

\[
p^D(q) = 10 - \frac{1}{2}q.
\]

The sets \( S \) and \( D \) are sketched in Figure 1.4.

A point \((q^*, p^*)\) lies in \( E = S \cap D \) if and only if the point satisfies both the equation of the supply set and the equation of the demand set. Thus, \((q^*, p^*)\) must be such that

\[
2p^* - 3q^* = 12
\]

and

\[
2p^* + q^* = 20.
\]

Subtracting the first equation from the second gives \(4q^* = 8\), or \(q^* = 2\). Substituting this value in the first equation gives \(2p^* - 6 = 12\), so that \(p^* = 9\). Thus \( E \) consists of the single point \((2, 9)\) and when the market is in equilibrium, the selling price will be 9 and the quantity sold will be 2.

**Example 1.3** Suppose that, in the market described in Example 1.2, an excise tax of 2 per unit is imposed. Determine the new equilibrium market price and quantity.

**Solution**

In the presence of an excise tax of 2, the effective price from the supplier’s point of view is not the market price, but the market price less 2. Thus, if \( p^T \) is the new equilibrium market price when the tax is imposed, then the quantity supplied, \( q^T \), satisfies both
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\[ q^T = q^S(p^T - 2) \quad \text{and} \quad q^T = q^D(p^T), \]

that is

\[ q^T = \frac{2}{3}(p^T - 2) - 4 \quad \text{and} \quad q^T = 20 - 2p^T. \]

Eliminating \( q^T \), we get

\[ \frac{2}{3}(p^T - 2) - 4 = 20 - 2p^T, \]

giving \( p^T = 19/2 \) and \( q^T = 20 - 2(19/2) = 1. \)

**Example 1.4** Suppose that the supply and demand functions for a good are

\[ q^S(p) = bp - a \quad \text{and} \quad q^D(p) = c - dp, \]

where \( a, b, c, d \) are positive constants. Show that the equilibrium price is \( p^* = (c + a)/(b + d) \). If an excise tax of \( T \) per unit is imposed (\( T \neq 0 \)), find the resulting market price \( p^T \) and show that \( p^T \) is strictly less than \( p^* + T \).

**Solution**

The equilibrium price \( p^* \) (in the absence of any tax) is found by solving the equations \( q^* = bp^* - a = c - dp^* \), which give \( p^* = (c + a)/(b + d) \).

When an excise tax of \( T \) per unit is imposed, the effective price from the supplier’s point of view is \( p^T - T \). The quantity sold, \( q^T \), satisfies

\[ q^T = b(p^T - T) - a \quad \text{and} \quad q^T = c - dp^T, \]

so that

\[ p^T = \frac{c + a}{b + d} + \left( \frac{b}{b + d} \right) T = p^* + \left( \frac{b}{b + d} \right) T. \]

Thus the selling price rises by \( (b/(b + d))T \) to its new equilibrium value. Since \( b \) and \( d \) are both positive, the fraction \( b/(b + d) \) is strictly less than 1, and hence the increase is strictly less than \( T \).

Note that we have verified mathematically a qualitative observation: in cases where the supply and demand sets are described by straight lines with upward and downward slopes, respectively, not all of an excise tax is passed on to the consumer. This is a case of what is often known as the **Tax Theorem**.

**Main Topics**

- interpretation of demand and supply sets
- demand, supply, inverse demand and inverse supply functions
- equilibrium price and quantity
- equilibrium price and quantity in the presence of excise tax
Key Terms, Notations and Formulae

- demand set, \( D \)
- supply set, \( S \)
- demand function, \( q^D(p) \); inverse demand function, \( p^D(q) \)
- supply function, \( q^S(p) \); inverse supply function, \( p^S(q) \)
- equilibrium set, \( E = S \cap D \); equilibrium quantity, \( q^* \) and price, \( p^* \)
- excise tax, \( T \); corresponding equilibrium quantity and price, \( q^T \) and \( p^T \)

Exercises

1.1 Suppose the supply and demand sets for Glenbowley single malt whisky are as follows: \( S \) consists of the pairs \((q, p)\) for which
\[
q - 3p = -5
\]
and \( D \) consists of the pairs \((q, p)\) for which
\[
q + 2p = 145.\]
Here, \( p \) is the price per bottle, measured in dollars, and \( q \) is the number of thousands of bottles sold. Determine the equilibrium price and quantity.

Suppose now that the government imposes an excise tax of \( T \) per bottle. What will be the new selling price and quantity sold?

1.2 The inverse supply and demand functions for a good are given by
\[
p^S(q) = 3q + 2 \quad \text{and} \quad p^D(q) = 6 - q,
\]
respectively. Find the equilibrium price and quantity.

Suppose that the government decides to impose an excise tax of \( T \) on each unit of this good. What price will the consumers end up paying for each unit of the good and how much will be sold?

Find a formula for the amount of money the government obtains from taxing this good.

1.3 The demand and supply functions for a good are, respectively,
\[
q^D(p) = 240 - 4p \quad \text{and} \quad q^S(p) = 8p.
\]
Find the equilibrium price and quantity.

A percentage sales tax of \( 100r\% \) is introduced (where \( 0 < r < 1 \)). Find the new equilibrium price and quantity.

Problems

1.1 If \( x + y = 3 \) and \( x - 2y = -3 \), what are \( x \) and \( y \)?

1.2 Solve the following simultaneous equations:
\[
2x + y = 9, \quad x - 3y = 1.
\]
1.3 Suppose the market for a commodity is governed by supply and demand sets defined as follows. The supply set $S$ is the set of pairs $(q,p)$ for which $q - 6p = -12$ and the demand set $D$ is the set of pairs $(q,p)$ for which $q + 2p = 40$.

Sketch $S$ and $D$ and determine the equilibrium set $E = S \cap D$, the supply and demand functions $q^S$ and $q^D$ and the inverse supply and demand functions $p^S$ and $p^D$.

1.4 Suppose that the government decides to impose an excise tax of $T$ on each unit of the commodity discussed in Problem 1.3. What price will the consumers end up paying for each unit of the commodity?

1.5 Find a formula for the amount of money the government obtains from taxing the commodity in the manner described in Problem 1.4. Determine this amount explicitly when $T = 0.5$.

1.6 The supply and demand functions for a commodity are

$$q^S(p) = 12p - 4 \quad \text{and} \quad q^D(p) = 8 - 4p.$$ 

If an excise tax of $T$ is imposed, what are the selling price and quantity sold, in equilibrium?