PART I

QUANTUM FIELD THEORY

Ouverture: Concepts of Quantum Field Theory

In this Ouverture, we anticipate fundamental concepts and basic principles of quantum field theory, in order to pave the way for a systematic exposition of the subject in the continuation of Part I of this book. In particular, we highlight the fundamental roles of *locality* and *symmetries*.

We emphasize, however, that understanding this introduction is by no means necessary for getting started with the book. Chapter 1 begins at a simple level, assuming only basic knowledge of quantum mechanics and special relativity. A reader who may be overwhelmed by the concepts mentioned in this Ouverture is encouraged to simply skip it and to proceed right away to Chapter 1. We recommend consulting this introductory overview repeatedly while working through Part I.

Point Particles versus Fields at the Classical Level

Theoretical physics in the modern sense was initiated by Isaac Newton who published his book Philosophiæ Naturalis Principia Mathematica in 1687. This eruption of genius provided us with the description of *classical point particle mechanics,* in terms of *ordinary* differential equations for the position vectors $\vec{x}_a(t)$ of N individual point particles ($a \in \{1, \ldots, N\}$) as functions of the time t. Classical mechanics is *local in time,* because Newton's equation contains time-derivatives, $d^2\vec{x}_a(t)/dt^2$, but no finite time-differences $t - t'$. On the other hand, Newtonian mechanics is *non-local in space,* because the finite distances $|\vec{x}_a - \vec{x}_b|$ between different particles determine instantaneous forces, in particular in Newtonian gravity. Hence in classical mechanics, there are conceptual differences between space and time. In point particle theories, the fundamental degrees of freedom – namely, the particle positions $\vec{x}_a(t)$ – are *mobile:* They roam around in space. As a consequence, at almost all points space is empty, *i.e.* nothing is there, except when a point particle occupies that position.

In contrast, the fundamental degrees of freedom of a *field theory* – the field values $\phi(\vec{x}, t)$ – are *immobile;* they are attached to a given space point $\vec{x}$ at all times t. In this case, it is the field value ϕ – not the position $\vec{x}$ – which changes as a function of time. In a field theory, space plays a role which is very different from point particle mechanics. In particular, it is *not empty* anywhere because field degrees of freedom exist at all points $\vec{x}$, at any time t. Fluid dynamics is an example for a non-relativistic classical field theory in which the mass density enters as a scalar field $\phi(\vec{x}, t)$. The classical field equations are *partial* differential equations – involving both space- and time-derivatives of $\phi(\vec{x}, t)$ – which

determine the evolution of the fields. Hence, in contrast to point particle theories, field theories are *local in both space and time*.

The most fundamental classical field theory is James Clerk Maxwell's electrodynamics of electric and magnetic fields, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ (Clerk Maxwell, 1865). Although this was not known until the end of the 19th century, Maxwell's electrodynamics is a *relativistic* classical field theory: It is invariant against space–time translations and rotations forming the *Poincaré symmetry group*. On the other hand, Newton's point particle mechanics is invariant under Galilean instead of Lorentz boosts. Thus, it is non-relativistic and therefore inconsistent with the relativistic Minkowski space–time underlying Maxwell's electrodynamics.

Albert Einstein's *special theory of relativity* (Einstein, 1905) modifies Newton's point particle mechanics in such a way that it becomes Poincaré-invariant. Indeed, in the framework of special relativity, charged point particles can interact with classical electromagnetic fields in a Poincaré-invariant manner. On the other hand, relativistic point particles cannot interact *directly* with each other; thus, they remain necessarily free in the absence of a mediating (*e.g.,* electromagnetic) field. This follows from Heinrich Leutwyler's *non-interaction theorem* for relativistic systems of N point particles (Leutwyler, 1965), which extends an earlier result for the 2-particle case (Currie *et al.*, 1963).

Indeed, in a relativistic quantum field theory, such as the Standard Model, the point particle concept is abandoned and all "particles" are just *field excitations,* which Frank Wilczek sometimes denotes as "wavicles" (Wilczek, 2012). This is a very useful distinction, which allows us to avoid confusion that might otherwise arise quite easily. In particular, while a Newtonian point particle has a completely well-defined position $\vec{x}_a$, a wavicle does not.

Particles versus Waves in Quantum Theory

Quantum mechanics, as formulated in the 1920s by Niels Bohr, Max Born, Werner Heisenberg, Erwin Schrödinger, and others (see Hund (1974) for a historic account), applies the basic principles of quantum theory – such as unitarity, which implies the conservation of probability – to Newton's point particles. As a consequence, the particle positions are affected by quantum uncertainty, but they still exist conceptually. This situation is described in terms of a wave function $\Psi(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N,t)$, which obeys a non-relativistic Schrödinger equation – a partial differential equation containing derivatives with respect to the time t and to the N particle positions $\vec{x}_a$. It is important to note that (unlike $\phi(\vec{x},t)$) $\Psi(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N,t)$ is *not* a field in space–time, but just a time-dependent complex function over the N-particle configuration space $(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N)$. A time-dependent state in a quantum field theory, on the other hand, can be described by a complex-valued wave functional $\Psi(\phi(\vec{x}),t) = \Psi([\phi],t)$, which depends on time and on the field configuration $[\phi]$ (at all space points) and which obeys a *functional* Schrödinger equation (Stückelberg, 1938; Tomonaga, 1946).

When one discusses quantum mechanical experiments of the *double-slit* type, one assigns the observed interference pattern to the wave properties of quantum particles. This

does, however, not mean that such a particle is understood as a quantized wave excitation of a field. It is just considered as a point particle (that is indeed endowed with the concept of a position), which is, however, affected by quantum uncertainty. As long as its position is not measured, it can "pass through both slits" (symbolically speaking), until it hits the detection screen that registers its position (which is then unambiguous, within the resolution of the detector). Only after repeating such a single-particle experiment a large number of times, the detected positions of the individual particles give rise to the well-known interference pattern. In the context of quantum mechanics, particle-wave duality just means that the spatial probability distribution of point particles results from a quantum mechanical wave function $\Psi(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N, t)$.

When a classical (*e.g.,* electromagnetic) wave is diffracted at a double slit, it shows an interference pattern for different reasons. As a field excitation, the wave exists simultaneously at all points in some region of space; it does not even conceptually have a sharp position. In contrast to the experiment with quantum mechanical point particles, the interference pattern arises immediately, as soon as the classical wave reaches the detection screen.

When one interprets this experiment with light at the quantum level and refers to individual photons, then the interference pattern again emerges only after the experiment has been repeated a large number of times. The complementary "particle" character of the photon is usually emphasized in the context of the Compton effect. However, while we may be used to thinking of an electron as a point particle (with some position, affected by quantum uncertainty), we should definitely not think of a photon in a similar way (and of the electron neither; see below). As a quantized wave excitation of the electromagnetic field, a photon does not even conceptually have a well-defined point-like position in space.

So what does it mean when we refer to the photon as a "particle"? Unfortunately, in our casual language the term particle is associated with the idea of a point-like (or small) object, which is not what the photon is like. Wilczek's term *wavicle* serves its purpose when it prevents us from thinking of a photon as a tiny ball. At the end of the day, only mathematics provides an appropriate and accurate description of "particles", like a photon. In the formalism of quantum field theory, "particle-wave duality" refers to the fact that "particles" actually are wavicles, *i.e.* quantized wave excitations of fields.

When Paul Adrien Maurice Dirac discovered his relativistic equation for the electron (Dirac, 1928), the 4-component *Dirac spinor* was initially interpreted as the wave function of an electron or a positron with spin up or down. However, relativity also requires the possibility of electron–positron pair creation at sufficiently high energy, which is indeed observed experimentally, along with pair annihilation. As a consequence, the Dirac equation does not have a consistent single-particle interpretation. Actually, the Dirac spinor is not a wave function, but a fermion field, whose quantized wave excitations manifest themselves as electrons or positrons (or other fermions). In other words, not only photons but *all elementary "particles"* are, in fact, wavicles. When Dirac's electron–positron field is coupled to the electromagnetic field, one arrives at Quantum Electrodynamics (QED), the quantized version of Maxwell's theory. Its construction was pioneered by Freeman Dyson, Richard Feynman, Julian Schwinger, and Sin-Itiro Tomonaga (Tomonaga, 1946; Schwinger, 1948; Dyson, 1949a,b; Feynman, 1949a,b). QED is an integral part of the Standard Model, in which all elementary "particles", including quarks, leptons, and the Higgs

particle, are quantized wave excitations of the corresponding fields. Unlike point particles, Higgs, quark, and lepton fields can interact in a relativistic manner, with or without the mediation by gauge fields.

Although in the Standard Model all "particles" are, in fact, wavicles, one often reads the statement that quarks and leptons are "point-like". What could this possibly mean for a wavicle that does not even have a conceptually well-defined position in space? Again, this is a deficiency of our casual language, which is properly resolved by the unambiguous mathematics of quantum field theory. What the above statement actually means is that even the highest energy experiments have, at least until now, not revealed any substructure of quarks or leptons; *i.e.* they seem to be truly elementary.[1] This does not apply to strongly interacting particles – known as hadrons – like protons, neutrons, or pions, which (essentially) consist of quarks, anti-quarks, and gluons. Interestingly, while being "point-like" in the above sense, an electron is at the same time "infinitely extended", due to the Coulomb cloud that surrounds it. Such "dressed" electrically charged particles are known as *infraparticles*. In the real world, however, their Coulomb field is usually screened by some opposite charge nearby.

This section should have convinced the reader that particle physics does *not* deal with point particles. Perhaps it would be better to call it "wavicle physics". However, as long as we are aware that our casual language is not sufficiently precise in this respect, the terminology is secondary. So, we will generally use the term "particle", keeping in mind that in quantum field theory this is actually a quantized, extended field excitation.

Classical and Quantum Gauge Fields

In the Standard Model, *gauge fields* play a key role (although it also contains non-gauge-field-mediated couplings between quark, lepton, and Higgs fields): Gauge fields mediate the fundamental strong, weak, and electromagnetic interactions. The (classical) Maxwell equations can be expressed entirely in terms of the electromagnetic field strengths $\vec{E}$ and $\vec{B}$, which form the anti-symmetric field strength tensor $F_{\mu\nu}$, with $F_{0i} = E_i$, $F_{ij} = -\epsilon_{ijk}B_k$. In relativistic quantum field theory, Abelian gauge fields are described by the 4-vector potential $A^\mu = (A^0, \vec{A})$. Together with the ordinary derivative ∂_μ, the 4-vector potential A_μ forms a *covariant derivative* $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$ (where e is the electric charge) and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In particular, the Aharonov–Bohm effect (Aharonov and Bohm, 1959, 1961; Berry, 1980; Peshkin and Tonomura, 1989) is naturally expressed through line integrals of the 3-vector potential $\vec{A}$. It is part of the 4-vector potential, which can be gauge transformed to $A_\mu{}' = A_\mu - \partial_\mu\alpha$, where $\alpha(\vec{x}, t)$ is an arbitrary (differentiable) gauge transformation function of space and time. In contrast to A_μ, the field strength tensor $F_{\mu\nu}$ is gauge-invariant and thus physical.

When we work with vector potentials, we employ redundant gauge-variant variables to describe gauge-invariant physical observables. While this is a matter of choice in classi-

[1] It could also be interpreted such that an elementary particle is always detected in a single pixel or cell of a detector, no matter how fine its resolution is. However, here enters the notorious quantum measurement problem, which we are not going to discuss.

cal theories, it seems inevitable in quantum theories. In particular, already in the quantum mechanics of a charged point particle, the complex phase ambiguity of the wave function turns into a local gauge freedom. Similarly, in quantum field theory the complex phase of a Dirac spinor field is gauge-*variant,* but it can be combined with the gauge-variant vector potential to form the gauge-*invariant* QED Lagrange density or *Lagrangian*. *Gauge invariance* is a local symmetry which must be maintained *exactly* in order to exclude unphysical effects due to the redundant gauge variables.

Since a gauge symmetry – which we consider synonymous to a *local symmetry* – just reflects a redundancy in our theoretical description of the gauge-invariant physics, it entails physical consequences which differ from those of a global symmetry. Both for global and for local symmetries, the Hamilton operator of the theory is invariant under symmetry transformations. In case of a *global* symmetry (at least in the absence of spontaneous symmetry breaking), this implies that energy eigenstates belong to (generally non-trivial) multiplets of the symmetry group. As a consequence, there are degeneracies in the spectrum whenever an irreducible representation is more than 1-dimensional. In case of a *gauge symmetry,* on the other hand, all physical states are gauge-invariant; *i.e.* they belong to a trivial 1-dimensional representation of the gauge group. Hence, gauge symmetries do not lead to degeneracies in the spectrum of physical states. Indeed, the gauge-variant eigenstates of the gauge-invariant Hamilton operator are exiled from the physical Hilbert space, by imposing the Gauss law as a constraint on physical states.

Ultraviolet Divergences, Regularization, and Renormalization

Field theories have a fixed number of fundamental field degrees of freedom attached to each point in space. In continuous space, the total number of field degrees of freedom is thus uncountably large. While this is not a problem in classical field theory, where the solutions of field equations are smooth functions of space and time, quantum fields perform fluctuations – even at infinitesimally close points – which give rise to ultraviolet (UV) divergences. In order to obtain meaningful finite values for physical quantities, quantum field theories must be *regularized*, for example, by introducing a UV cut-off.

One may speculate that regularization would no longer be necessary if one could work directly with the ultimate degrees of freedom that may exist at ultra-short distances, say of the order of the *Planck length* $l_{\text{Planck}} \simeq 1.62 \cdot 10^{-35}$ m. The corresponding energy scale is the *Planck mass* $M_{\text{Planck}} = \hbar/(c\, l_{\text{Planck}}) \simeq 1.22 \cdot 10^{19}$ GeV, at which gravity (which is extremely weak at low energies) is expected to become strongly coupled. Although string theory provides a framework for its formulation, an established non-perturbative theory of quantum gravity, valid even at the Planck scale, is not known until now. The necessity for regularization apparently suggests that quantum fields in continuous space–time may ultimately not be the correct degrees of freedom that Nature is built of at ultra-short distances.

Fortunately, we do not need to know the *Theory of Everything* before we can address the physics in the TeV energy regime that is accessible to present-day accelerator experiments, which have tested the Standard Model with great scrutiny. The currently accessible TeV

energy physics is insensitive to the details of more fundamental, underlying degrees of freedom, be they strings or something else.

In order to mimic the effects of the unknown, ultimate, ultra-short distance degrees of freedom, one can introduce a regularization in different ways. It is crucial that the regularization does not violate important symmetries, in particular, gauge symmetries; otherwise, it is hard to restore them in the final limit, and unphysical redundant variables may contaminate the result. In perturbation theory, the most efficient way to introduce a gauge-invariant regularization is *dimensional regularization*, *i.e.* analytic continuation in the space–time dimension d (Bollini and Giambiagi, 1972a,b; 't Hooft and Veltman, 1972). Dimensional regularization actually leads to finite perturbative results without explicitly introducing a UV cut-off.

Beyond perturbation theory, the *lattice regularization,* which reduces space–time to a 4-dimensional (usually hyper-cubic) grid of discrete lattice points, provides a natural cut-off that allows us to maintain gauge invariance (Wilson, 1974); for lattice textbooks, see Creutz (1983); Rothe (1992); Montvay and Münster (1994); Smit (2002); DeGrand and DeTar (2006); Gattringer and Lang (2009); Knechtli *et al.* (2017). In this case, the lattice spacing a, *i.e.* the distance between nearest-neighbor lattice sites, acts as an inverse UV cut-off π/a. In contrast to continuous space–time, in lattice field theory the number of degrees of freedom becomes *countable,* which removes the divergences in physical observables. Still, in order to obtain meaningful physical results, one must extrapolate to the *continuum limit,* $a \to 0$. This is achieved by tuning the coupling constants in the Lagrangian in such a way that the long-distance continuum physics becomes insensitive to the lattice spacing. Such systems can be described using the terminology of classical statistical mechanics, where the continuum limit corresponds to a *critical point*. Approaching criticality is related to the process of *renormalization.*

Euclidean Quantum Field Theory versus Classical Statistical Mechanics

The quantization of field theories is a subtle mathematical problem. The functional integral approach (*i.e.* Feynman's path integral applied to field theory) offers an attractive alternative to canonical quantization. When real Minkowski time is analytically continued to purely imaginary Euclidean time, the functional integral becomes mathematically much better behaved. As an extra benefit, Euclidean quantum field theory, in particular when formulated on a space–time lattice, is mathematically analogous to a system of classical statistical mechanics. The Euclidean fields correspond to generalized spin variables, and the classical Hamilton function is analogous to the Euclidean lattice action of quantum field theory. The temperature T, which controls the thermal fluctuations in statistical mechanics, is analogous to $\hbar$, which controls the strength of quantum fluctuations. A spin correlation function is analogous to a Euclidean 2-point function. Usually, it exhibits an exponential decay, which determines a *correlation length* $\xi = \hbar/(Mc)$, where M is a particle mass.

The analogy between classical statistical mechanics and Euclidean quantum field theory has far-reaching consequences, because the theory of *critical phenomena* can be translated to quantum field theory. In particular, a critical point, where the correlation length diverges

in units of the lattice spacing, $\xi/a \to \infty$, corresponds to the continuum limit of a Euclidean lattice field theory with $Ma \to 0$. The insensitivity of low-energy physics to the details of the regularization of quantum field theory is a manifestation of *universality*. *Relevant, marginal,* and *irrelevant* couplings are identified based on the *renormalization group* (Wilson and Kogut, 1974; Wilson, 1975). Furthermore, Monte Carlo methods, which were originally developed for classical statistical mechanics, can be applied to lattice field theory where they provide systematically controlled, non-perturbative results.

1 Basics of Quantum Field Theory

This chapter presents an introduction to quantum field theory in the *functional integral formalism.* Classical field theories are introduced as an extension of point particle mechanics to systems with infinitely many degrees of freedom – a fixed number at each space point. Similarly, quantum field theories formally correspond to quantum mechanical systems with infinitely many degrees of freedom. In the same way as point particle mechanics, classical field theories can be quantized by means of the path integral – or *functional integral* – method. A schematic overview is sketched in Figure 1.1.

The transition from the physical real time to Euclidean time, by means of a *Wick rotation,* is highly favorable for the convergence of functional integrals. The resulting quantum field theories in Euclidean space–time can be related to classical statistical mechanics. In this context, we also address the *lattice regularization,* which provides a formulation of quantum field theories beyond perturbation theory.

1.1 From Point Particle Mechanics to Classical Field Theory

Point particle mechanics describes the dynamics of classical, non-relativistic point particles. The coordinates of the particles represent a finite number of degrees of freedom. In the simplest case – a single particle moving on a line – this degree of freedom is just given by the particle position[1] x, as a function of the time t. The dynamics of a particle of mass M moving in an external potential $V(x)$ obeys *Newton's equation of motion*

$$M\partial_t^2 x = F(x) = -\frac{dV(x)}{dx}. \tag{1.1}$$

Once the initial conditions are specified, this ordinary second-order differential equation determines the path of the particle, $x(t)$. To compute it, one proceeds in infinitesimal time steps.

An alternative approach considers a *finite* time interval with a given initial and final particle position and identifies the classical path connecting them. In this setting, Newton's equation is obtained from the *variational principle* by minimizing the *action,*

$$S[x] = \int dt\, L(x, \partial_t x), \tag{1.2}$$

[1] For the considerations here, and in Sections 1.2 and 1.3, the space dimension hardly matters. For simplicity we set it to 1, but a generalization to higher dimensions is straightforward; one just replaces $x(t)$ by $\vec{x}(t)$ everywhere.

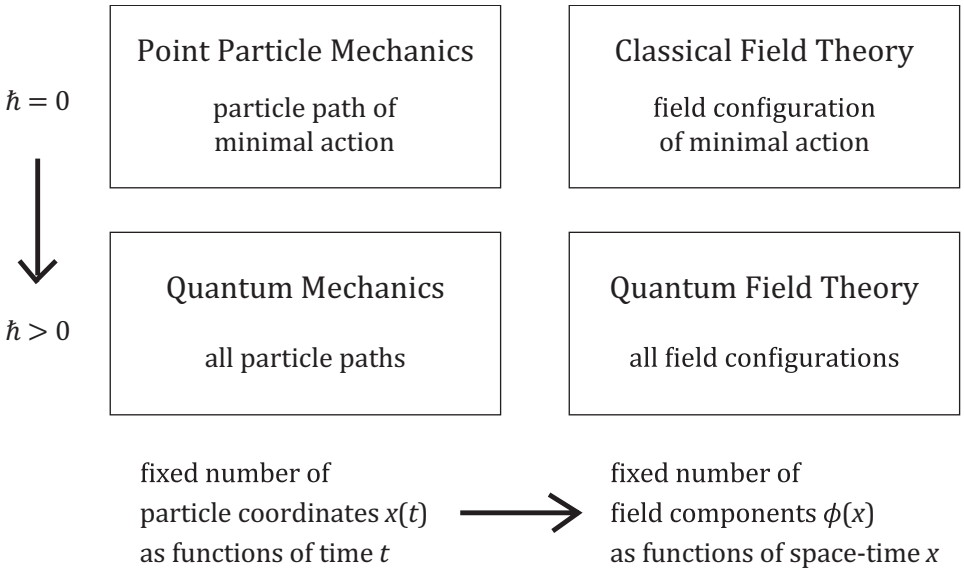

Fig. 1.1 Overview of different branches of physics: We proceed from point particle mechanics to field theory (left to right) and from classical physics to quantum physics (top to bottom).

in the set of all paths $[x]$ with the given end-points.[2] The action is a functional[3] that results from the time-integral of the *Lagrange function*

$$L(x, \partial_t x) = \frac{M}{2}(\partial_t x)^2 - V(x) \tag{1.3}$$

over a path with fixed end-points in space and time. Now the variational condition $\delta S = 0$ implies the *Euler–Lagrange equation*

$$\partial_t \frac{\delta L}{\delta \partial_t x} - \frac{\delta L}{\delta x} = 0, \tag{1.4}$$

which coincides with Newton's equation (1.1) at any instant t.

Classical field theories are a generalization of point particle mechanics to systems with infinitely many degrees of freedom – a fixed number N at each space point $\vec{x}$. Here the degrees of freedom are the field values $\phi_a(\vec{x}, t)$, $a \in \{1, 2, \ldots, N\}$, where ϕ represents a generic field. We anticipate a few examples:

- In the case of a *neutral scalar field*, the value of ϕ is a single real number, $\phi \in \mathbb{R}$, representing $N = 1$ degree of freedom per space point.
- A *charged scalar field* is described by a complex number, $\Phi = \phi_1 + \mathrm{i}\phi_2 \in \mathbb{C}$ (with $\phi_i \in \mathbb{R}$). Hence, it represents $N = 2$ real degrees of freedom per space point.
- The *Higgs field* $\phi_a(\vec{x}, t)$ $(a \in \{1, \ldots, 4\})$, which is part of the Standard Model, is a complex doublet; it has $N = 4$ real degrees of freedom per space point.

[2] More precisely, one identifies a stationary point in the set of possible paths connecting fixed end-points.
[3] We use the term "functional" for a function whose argument is itself a function, which is integrated (or summed) over. Hence the functional depends on the entire function in its argument, not just on its value at a specific point; the latter is the case for the Lagrange function $L(x, \partial_t x)$.