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I dedicate this book to my family, especially my wife Priscilla and
my daughter Ella. I also dedicate this book to God,
without whom this would not have been possible.

Contents

Preface

In this book we study the semilinear Schrödinger equation with power-type nonlinearity. This equation has been an active area of research in dispersive partial differential equations. The study of semilinear Schrödinger equations is useful in its own right, since it has many applications to physics. It also provides a great deal of insight into other dispersive partial differential equations and geometric partial differential equations. The study of this equation combines tools from harmonic analysis, microlocal analysis, functional analysis, and topology. It is a truly fascinating topic.

This book is mainly focused on the mass-critical (or L^2-critical) problem with initial data in L^2, and the energy-critical (or $\dot{H}^1$-critical) problem with initial data in $\dot{H}^1$. These problems have been shown to be globally well posed and scattering in the defocusing case for critical initial data, and moreover, these results are sharp. Presentation of the proofs of these results is the goal of this book.

Chapter 1 commences the study of the mass-critical problem. A natural approach to a nonlinear problem is to treat it as a perturbation of a linear problem. Thus, Chapter 1 begins with an examination of the dispersive properties of solutions to the linear equation. In the section that follows, the dispersive estimates are utilized to prove Strichartz estimates for the linear equation. These estimates are very important since they are invariant under time translation of the linear solution operator. In Section 1.3 these estimates are used to prove that the mass-critical problem is scattering when the mass is small. The argument utilizes a standard fixed point argument. Chapter 1 then concludes with a proof of scattering for solutions to the mass-critical problem for data lying in a subspace of $L^2\left(\mathbf{R}^d\right)$, but with no size restriction on the initial data. The proof combines conserved quantities with perturbative arguments.

Chapter 2 addresses the cubic nonlinear Schrödinger equation in dimensions three and four, where it is mass supercritical, and either energy subcritical ($d = 3$) or energy critical ($d = 4$). There the small data problem is complicated by the need to differentiate the nonlinearity, especially in three dimensions, where the fractional product rule is used. However, the fact that the nonlinearity is cubic makes the analysis far more tractable than the analysis of mass-supercritical problems in higher dimensions. In Section 2.2 the three-dimensional cubic problem is discussed. This problem is $\dot{H}^{1/2}$-critical, but study of this problem has provided a great deal of insight into both the mass-critical and energy-critical problems (see for example Kenig and Merle (2010) and Lin and Strauss (1978)). In this section a Morawetz estimate of Lin and Strauss (1978) is proved, directly yielding a scattering result. In the final section of Chapter 2, the scattering result of Bourgain (1999) is presented. This work is widely considered to be the seminal paper for the techniques discussed in this book, although Grillakis (2000) contemporaneously proved global well-posedness in three dimensions for radial initial data. Bourgain (1999) actually proved scattering for the energy-critical, radially symmetric problem in dimensions $d = 3$ and 4. In keeping with the theme of cubic problems for this chapter, only the $d = 4$ result is discussed. The three-dimensional proof may be proved using the same argument as the four-dimensional proof, with only minor modifications.

The energy-critical problem in higher dimensions is addressed in Chapter 3. The first order of business is to prove small data scattering in dimensions $d \geq 5$. Thus, the perturbative energy-critical results of Tao and Visan (2005) are presented in the first section. Section 3.2 discusses Keraani's's (2001) profile decomposition for the energy-critical problem, commencing the discussion of the concentration compactness method with regard to the Schrödinger equation. In Section 3.3 the profile decomposition is used to prove that the defocusing, energy-critical nonlinear Schrödinger equation is scattering in dimensions $d \geq 5$. This result was originally proved in Visan (2007) (see also Visan (2006)). However, in this book the result is proved using a slight modification of the argument in Killip and Visan (2010). The argument utilizes the double Duhamel argument and the interaction Morawetz estimate. The interaction Morawetz estimates of Colliander *et al*. (2004) and Tao *et al*. (2007a) are proved in Section 3.4.

Higher-dimensional mass-critical results are presented in Chapter 4. Bilinear estimates are essential to the study of the mass-critical problem. The interaction Morawetz estimates lend themselves quite well to bilinear estimates, a fact that was well exploited by Planchon and Vega (2009). Section 4.1 discusses both the Fourier analytic approach to bilinear estimates found in

Bourgain (1998), as well as the interaction Morawetz approach of Planchon and Vega (2009). Both approaches are used to study the mass-critical problem in Chapters 4 and 5. The subsequent section presents the mass-critical profile decomposition of Tao *et al*. (2008). This profile decomposition crucially relies on the bilinear estimates of Tao (2003), which will not be proved in this book. Then Section 4.3 presents the scattering results of Killip *et al*. (2009), Killip *et al*. (2008), and Tao *et al*. (2007b) for radial data. Section 4.4 then presents the scattering result of Dodson (2012) for nonradial data.

Chapter 5 addresses the defocusing, energy-critical and mass-critical problems in low dimensions. The scattering results of Colliander *et al*. (2008) ($d = 3$) and Ryckman and Visan (2007) ($d = 4$) for the defocusing, energy-critical problem are presented in the first two sections. The $d = 4$ case is completely worked out in the first section, while some of the more technically difficult parts of the $d = 3$ case are presented in Section 5.2. Sections 5.3 and 5.4 follow a similar pattern for the mass-critical problem. There the scattering results of Dodson (2016a) ($d = 1$) and Dodson (2016b) ($d = 2$) are presented for the defocusing, mass-critical problem. The one-dimensional result of Dodson (2016a) is completely worked out in Section 5.3, while some of the technically difficult aspects of the two-dimensional result are resolved in Section 5.4.

In the author's opinion the material would be sufficient for a one-semester course for graduate students who have taken a course in real analysis at the graduate level. The author assumes that the reader is familiar with basic measure theory and integration, such as can be found in Lieb and Loss (2001) or Taylor (2006). The author also assumes that the reader is familiar with basic functional analysis concepts such as Banach spaces, Hölder spaces, and Fréchet spaces. Conway (1990) and Yosida (1980) provide good overall introductions to the field of functional analysis. Finally, the book uses many techniques from the field of harmonic analysis such as interpolations and stationary phase analysis. Grafakos (2004), Muscalu and Schlag (2013), Sogge (1993), and Stein (1993) provide good introductions to harmonic analysis.

The term "nonlinear Schrodinger equation" will often by abbreviated NLS.

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