1

A First Look at the Mass-Critical Problem

1.1 Linear Schrödinger Equation and Preliminaries

Formally, the solution to the linear Schrödinger equation,

$$iu_t + \Delta u = 0, \quad u(0,x) = u_0, \tag{1.1}$$

may be given by $e^{it\Delta}u_0$. Here Δ is a self-adjoint operator on any L^2-based Sobolev space, so $i\Delta$ is skew-adjoint on any L^2-based Sobolev space, and thus the operator $e^{it\Delta}$ is a perfectly well-defined unitary group. (See Section A.9 of Taylor (2011) for a proper introduction to unitary groups.)

When viewed from a Fourier-analytic perspective, the spectral theory of (1.1), and thus the operator $e^{it\Delta}$, is readily apparent.

Definition 1.1 (Fourier transform) For $f \in L^1\left(\mathbf{R}^d\right)$ let

$$(\mathscr{F} f)(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} f(x)\, dx.$$

The Fourier transform intertwines translation and multiplication. Let T_{x_0} be the translation operator

$$T_{x_0} f = f(x - x_0).$$

Then by a change of variables, for $f \in L^1\left(\mathbf{R}^d\right)$,

$$\mathscr{F}\left(T_{x_0} f\right)(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} f(x - x_0)\, dx = e^{-ix_0\cdot\xi} \hat{f}(\xi) \tag{1.2}$$

and

$$\begin{aligned} T_{\xi_0} \mathscr{F}(f)(\xi) &= \mathscr{F}(f)(\xi - \xi_0) = (2\pi)^{-d/2} \int e^{-ix\cdot(\xi - \xi_0)} f(x)\, dx \\ &= \mathscr{F}\left(e^{ix\cdot\xi_0} f\right)(\xi). \end{aligned} \tag{1.3}$$

Equations (1.2) and (1.3) are formally equivalent to

$$\mathscr{F}(\partial_x^\alpha f)(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} (\partial_x^\alpha f(x))\, dx$$
$$= (2\pi)^{-d/2} \int e^{-ix\cdot\xi} (i\xi)^\alpha \hat{f}(\xi)\, d\xi \;= (i\xi)^\alpha \hat{f}(\xi) \qquad (1.4)$$

and

$$\mathscr{F}(x^\alpha f)(\xi) = (2\pi)^{-d/2} \int x^\alpha e^{-ix\cdot\xi} f(x)\, dx$$
$$= (2\pi)^{-d/2} i^\alpha \int \partial_\xi^\alpha \left(e^{-ix\cdot\xi}\right) f(x)\, dx$$
$$= \left(i\partial_\xi\right)^\alpha \hat{f}(\xi), \qquad (1.5)$$

respectively, where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, $\alpha_i \geq 0$ for all $1 \leq i \leq d$.

Thus if $u(t,x)$ solves (1.1), then the Fourier transform of u (formally) solves

$$i\partial_t(\hat{u}(t,\xi)) - |\xi|^2 \hat{u}(t,\xi) = 0, \quad \hat{u}(0,\xi) = \hat{u}_0(\xi), \qquad (1.6)$$

so for each $\xi \in \mathbf{R}^d$, (1.6) gives an ordinary differential equation in time whose solution is

$$\hat{u}(t,\xi) = e^{-it|\xi|^2} \hat{u}_0(\xi). \qquad (1.7)$$

It is possible to formally compute $u(t,x)$ from $\hat{u}(t,\xi)$ using the inverse Fourier transform.

Definition 1.2 (Inverse Fourier transform) If $g \in L^1\left(\mathbf{R}^d\right)$, let

$$\left(\mathscr{F}^{-1} g\right)(x) = \check{g}(x) = (2\pi)^{-d/2} \int g(\xi) e^{ix\cdot\xi} d\xi.$$

The quantity $\mathscr{F}^{-1}\mathscr{F}$ is the identity on a set of functions that is dense in $L^p\left(\mathbf{R}^d\right)$ for any $1 \leq p < \infty$.

Definition 1.3 (Schwartz space) Let $\mathscr{S}\left(\mathbf{R}^d\right)$ be the set of functions such that

$$\mathscr{S}\left(\mathbf{R}^d\right) = \left\{ f \in C^\infty\left(\mathbf{R}^d\right) : \sup_{x\in\mathbf{R}^d} |x|^\beta |\partial_x^\alpha f| \leq C(\alpha,\beta) < \infty \; \forall \alpha,\beta \in (\mathbf{Z}_{\geq 0})^d \right\}. \qquad (1.8)$$

Remark The Schwartz space of smooth functions is actually a Fréchet space (but not a Banach space).

Clearly $\mathscr{S}\left(\mathbf{R}^d\right) \subset L^\infty\left(\mathbf{R}^d\right)$, and since $\left(1+|x|^2\right)^{-d}$ is integrable on $\mathbf{R}^d$, $\mathscr{S}\left(\mathbf{R}^d\right) \subset L^1\left(\mathbf{R}^d\right)$, so $\mathscr{F}$ is well defined on $\mathscr{S}\left(\mathbf{R}^d\right)$. Furthermore, (1.4) and (1.5) imply

$$\mathscr{F}, \mathscr{F}^{-1} : \mathscr{S}\left(\mathbf{R}^d\right) \to \mathscr{S}\left(\mathbf{R}^d\right),$$

so $\mathscr{F}^{-1}\mathscr{F}$ is well defined on $\mathscr{S}\left(\mathbf{R}^d\right)$.

Lemma 1.4 (Fourier inversion for Schwartz functions) *If* $f \in \mathscr{S}\left(\mathbf{R}^d\right)$, *then*

$$f(x) = \left(\mathscr{F}^{-1}\hat{f}\right)(x) = (2\pi)^{-d/2} \int e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

Proof For any $\hat{f}(\xi) \in \mathscr{S}\left(\mathbf{R}^d\right)$, a direct calculation shows that

$$(2\pi)^{-d/2} \int e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi = (2\pi)^{-d/2} \lim_{\varepsilon \searrow 0} \int e^{-\varepsilon|\xi|^2} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \tag{1.9}$$

uniformly in x. Since f is integrable, Fubini's theorem implies that for any $\varepsilon > 0$,

$$\begin{aligned}(1.9) &= (2\pi)^{-d} \int \int e^{i(x-y)\cdot\xi} e^{-\varepsilon|\xi|^2} f(y)\, dy\, d\xi \\ &= (2\pi)^{-d} \int f(y) \int e^{i(x-y)\cdot\xi} e^{-\varepsilon|\xi|^2} d\xi\, dy \\ &= (4\pi\varepsilon)^{-d/2} \int f(y)\, e^{-\frac{|x-y|^2}{4\varepsilon}} dy. \end{aligned} \tag{1.10}$$

The last equality follows from the computation

$$\left(\int e^{-x^2} dx\right)^2 = \int e^{-x^2-y^2} dx\, dy = 2\pi \int_0^\infty e^{-r^2} r\, dr = \pi,$$

and completing the square in the exponent. A change of variables implies that for any $\varepsilon > 0$, $x \in \mathbf{R}^d$,

$$(4\pi\varepsilon)^{-d/2} \int e^{-\frac{|x-y|^2}{4\varepsilon}} dy = 1,$$

so

$$\lim_{\varepsilon \searrow 0} (4\pi\varepsilon)^{-d/2} e^{-\frac{|x-y|^2}{4\varepsilon}} = \delta(y-x).$$

For any $x \in \mathbf{R}^d$, $\delta(y-x)$ is a tempered distribution, and therefore if $f \in \mathscr{S}\left(\mathbf{R}^d\right)$, for any $x \in \mathbf{R}^d$,

$$\lim_{\varepsilon \searrow 0} (4\pi\varepsilon)^{-d/2} \int f(y)\, e^{-\frac{|x-y|^2}{4\varepsilon}} dy = f(x).$$

Further computations also show that this convergence is uniform in $\mathbf{R}^d$, and is uniform in all the seminorms in (1.8). Thus, $\mathscr{F}^{-1}\mathscr{F}$ is the identity on $\mathscr{S}\left(\mathbf{R}^d\right)$. A similar argument shows that $\mathscr{F}\mathscr{F}^{-1}$ is the identity on $\mathscr{S}\left(\mathbf{R}^d\right)$. □

Therefore, the solution to (1.1), formally given by

$$\mathscr{F}^{-1}\left(e^{-it|\xi|^2} \hat{u}_0(\xi)\right), \tag{1.11}$$

is well defined for any $t \in \mathbf{R}$ when u_0 is a Schwartz function.

Let $\langle \cdot, \cdot \rangle$ denote the L^2 inner product of functions,

$$\langle f, g \rangle = \text{Re} \int f(x) \overline{g(x)} dx.$$

For any f, $g \in \mathscr{S}(\mathbf{R}^d)$,

$$\begin{aligned} \langle \mathscr{F} f, g \rangle &= (2\pi)^{-d/2} \text{Re} \int \overline{g(\xi)} \int e^{-ix\cdot\xi} f(x)\, dx\, d\xi \\ &= (2\pi)^{-d/2} \text{Re} \int f(x) \left(\overline{\int e^{ix\cdot\xi} g(\xi)\, d\xi} \right) dx \\ &= \langle f, \mathscr{F}^{-1} g \rangle. \end{aligned} \tag{1.12}$$

Taking $g = \mathscr{F} f$, $f \in \mathscr{S}(\mathbf{R}^d)$ implies

$$\|\mathscr{F} f\|_{L^2(\mathbf{R}^d)} = \|f\|_{L^2(\mathbf{R}^d)}. \tag{1.13}$$

Since $\mathscr{S}(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}^d)$, (1.12) may be extended to the Parseval identity,

$$\langle \mathscr{F} f, g \rangle = \langle f, \mathscr{F}^{-1} g \rangle \quad \text{for any } \; f, g \in L^2(\mathbf{R}^d),$$

and (1.13) may be extended to the Plancherel identity,

$$\|f\|_{L^2(\mathbf{R}^d)} = \|\mathscr{F} f\|_{L^2(\mathbf{R}^d)} \quad \text{for any } \; f \in L^2(\mathbf{R}^d). \tag{1.14}$$

Since $|e^{it|\xi|^2}| = 1$, (1.7) and (1.14) imply that if u solves (1.1), then

$$\|u(t,x)\|_{L^2(\mathbf{R}^d)} = \left\| e^{it|\xi|^2} \hat{u}_0(\xi) \right\|_{L^2(\mathbf{R}^d)} = \|\hat{u}_0(\xi)\|_{L^2(\mathbf{R}^d)} = \|u_0\|_{L^2(\mathbf{R}^d)}. \tag{1.15}$$

Moreover, since $\mathscr{S}(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}^d)$, (1.15) implies that (1.11) is well defined for any $u_0 \in L^2(\mathbf{R}^d)$.

Next, using the Fourier and inverse Fourier transforms, if $u_0 \in \mathscr{S}(\mathbf{R}^d)$,

$$\begin{aligned} u(t,x) &= (2\pi)^{-d} \lim_{\varepsilon \searrow 0} \int e^{-\varepsilon|\xi|^2} e^{-it|\xi|^2} e^{ix\cdot\xi} \int e^{-iy\cdot\xi} u_0(y)\, dy\, d\xi \\ &= (2\pi)^{-d} \lim_{\varepsilon \searrow 0} \int \left(\int e^{-\varepsilon|\xi|^2} e^{-it|\xi|^2} e^{i(x-y)\cdot\xi} d\xi \right) u_0(y)\, dy \\ &= \lim_{\varepsilon \searrow 0} \int K(t, x-y, \varepsilon) u_0(y)\, dy. \end{aligned} \tag{1.16}$$

Completing the square in the exponent,

$$K(t,x-y,\varepsilon) = (2\pi)^{-d}\int e^{-\varepsilon|\xi|^2}e^{-it|\xi|^2}e^{i(x-y)\cdot\xi}d\xi$$
$$= (2\pi)^{-d}\int e^{-\varepsilon|\xi|^2}e^{-it|\xi-\frac{x-y}{2t}|^2}e^{i\frac{|x-y|^2}{4t}}d\xi, \tag{1.17}$$

so by stationary phase analysis (see Chapter 8 of Stein (1993)), letting $\varepsilon \searrow 0$,

$$u(t,x) = (4\pi t)^{-d/2}e^{-i\frac{d\pi}{4}}\int e^{i\frac{|x-y|^2}{4t}}u_0(y)\,dy. \tag{1.18}$$

Again, since $\mathscr{S}(\mathbf{R}^d)$ is dense in $L^p(\mathbf{R}^d)$, for any $1 \leq p < \infty$, (1.18) gives the dispersive estimate

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbf{R}^d)} \leq \frac{1}{|4\pi t|^{d/2}}\|u_0\|_{L^1(\mathbf{R}^d)}. \tag{1.19}$$

Then by (1.15), (1.19), and the Riesz–Thorin interpolation theorem (see for example Bergh and Löfstrom (1976)), for $2 \leq p \leq \infty$, if p' is the Lebesgue dual exponent satisfying $\frac{1}{p}+\frac{1}{p'}=1$,

$$\|e^{it\Delta}u_0\|_{L^p(\mathbf{R}^d)} \leq \frac{1}{|4\pi t|^{d(1/2-1/p)}}\|u_0\|_{L^{p'}(\mathbf{R}^d)}. \tag{1.20}$$

Therefore, for any $u_0 \in \mathscr{S}(\mathbf{R}^d)$,

$$\int_{\mathbf{R}^d}|u(t,x)|^2dx$$

remains constant, while at the same time,

$$\sup_{x\in\mathbf{R}^d}|u(t,x)|^2 \to 0$$

as $t \to \pm\infty$. Thus $|u(t,x)|^2$ spreads out as $t \to \pm\infty$. For this reason (1.19) is called a dispersive estimate and (1.1) is called a dispersive equation.

The reader who is familiar with the wave equation should observe that the dispersion in (1.19) is faster by a factor of $t^{-1/2}$ than for the wave equation in the same dimension. For example, consider the wave equation in one dimension, and choose some $f \in \mathscr{S}(\mathbf{R})$. The solution to

$$u_{tt}-\Delta u = 0, \qquad u(0,x) = f(x), \qquad u_t(0,x) = 0 \tag{1.21}$$

is given by

$$u(t,x) = \frac{1}{2}f(x+t)+\frac{1}{2}f(x-t).$$

Such a solution consists of two traveling waves, and thus does not disperse at all.

On the other hand, the solution to the linear Schrödinger equation with $u_0 = f$ will have L^∞-norm that decays at the rate of $t^{-1/2}$. The reason for this difference is that the wave equation obeys finite propagation speed and the Huygens principle. Thus, a solution to the one-dimensional problem (1.21) cannot disperse at all, since it travels at either speed 1 to the right or speed 1 to the left. For the linear Schrödinger equation, following (1.16), and formally taking $\varepsilon \searrow 0$,

$$(2\pi)^{-d} \int e^{-it|\xi|^2} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi = (2\pi)^{-d} \int e^{-it|\xi - \frac{x}{2t}|^2} e^{i\frac{|x|^2}{4t}} \hat{f}(\xi)\, d\xi,$$

and thus formally $\hat{f}(\xi)$ travels with velocity 2ξ. This computation can be made rigorous using the Littlewood–Paley decomposition.

Definition 1.5 (Littlewood–Paley decomposition) Let $\psi(\xi)$ be a smooth, decreasing, radial function supported on $|\xi| \leq 2$, $\psi(\xi) = 1$ on $|\xi| \leq 1$, and let

$$\phi_k(\xi) = \psi(2^{-k-1}\xi) - \psi(2^{-k}\xi). \tag{1.22}$$

Then $\phi_k(\xi)$ is a radial, smooth function supported on the annulus $2^k \leq |\xi| \leq 2^{k+2}$. Also, for $\xi \neq 0$,

$$\sum_{k=-\infty}^{\infty} \phi_k(\xi) = 1. \tag{1.23}$$

Let P_k be the Fourier multiplier given by $\phi_k(\xi)$; that is,

$$P_k f = \mathscr{F}^{-1}\left(\phi_k(\xi)\hat{f}(\xi)\right). \tag{1.24}$$

Also let

$\widetilde{P}_k = P_{k-2} + P_{k-1} + P_k + P_{k+1} + P_{k+2}$ and $\tilde{\phi}_k = \phi_{k-2} + \phi_{k-1} + \phi_k + \phi_{k+1} + \phi_{k+2}$.

Since $\phi_k(\xi)$ is supported on $2^k \leq |\xi| \leq 2^{k+2}$, (1.23) implies

$$\widetilde{P}_k P_k = P_k.$$

Define

$$P_{\leq k} = \sum_{j \leq k} P_j, \quad P_{<k} = \sum_{j<k} P_j, \quad P_{\geq k} = \sum_{j \geq k} P_j, \quad P_{>k} = \sum_{j>k} P_j.$$

Also, for any $N > 0$, define

$$\begin{aligned} (P_N f)(x) &= \mathscr{F}^{-1}\left(\phi_0\left(\frac{\xi}{N}\right)\hat{f}(\xi)\right), \\ (P_{<N} f)(x) &= \mathscr{F}^{-1}\left(\psi\left(\frac{\xi}{N}\right)\hat{f}(\xi)\right), \\ (P_{\geq N} f)(x) &= \mathscr{F}^{-1}\left(\left(1 - \psi\left(\frac{\xi}{N}\right)\right)\hat{f}(\xi)\right). \end{aligned} \tag{1.25}$$

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More Information

When there is no confusion it is convenient to write $f_k = P_k f$, $f_N = P_N f$, $f_{\leq N} = P_{\leq N} f$, $f_{\geq N} = P_{\geq N} f$. In this book, a lowercase Latin letter always refers to the projection (1.24) and an uppercase Latin letter always refers to the projection (1.25).

Theorem 1.6 (Littlewood–Paley theorem) *For any* $1 < p < \infty$,

$$\|f\|_{L^p(\mathbf{R}^d)} \sim_{p,d} \left\| \left(\sum_{j \in \mathbf{Z}} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^d)}. \tag{1.26}$$

Proof See Stein (1970). □

A solution to (1.1) that is localized to frequencies $|\xi| \sim 2^k$ travels at speed $\sim 2^k$.

Theorem 1.7 *Let K_k be the kernel of $P_k e^{it\Delta}$, where P_k is the Littlewood–Paley projection defined in Definition* 1.5. *That is, as in* (1.11) *and* (1.16), *let*

$$u(t,x) = e^{it\Delta} P_k u_0(x) = \int K_k(t, x-y)\, u_0(y)\, dy,$$

where

$$K_k(t,x) = (2\pi)^{-d} \int e^{ix\cdot\xi} e^{-it|\xi|^2} \phi_k(\xi)\, d\xi. \tag{1.27}$$

If $|x| \leq 2^{k+4}|t|$,

$$|K_k(t,x)| \lesssim_d \frac{2^{dk}}{(1 + 2^{2k}|t|)^{d/2}}. \tag{1.28}$$

If $|x| > 2^{k+4}|t|$, *then for any* M,

$$|K_k(t,x)| \lesssim_{d,M} \frac{2^{dk}}{(1 + 2^k|x|)^M}, \tag{1.29}$$

and if $|x| < 2^{k-4}|t|$, *then for any* M,

$$|K_k(t,x)| \lesssim_{d,M} \frac{2^{dk}}{(1 + 2^{2k}|t|)^M}. \tag{1.30}$$

Remark The kernel for the Littlewood–Paley projection operator is given by $K_j(0,x)$, where

$$P_j f(x) = \int K_j(0, x-y)\, f(y)\, dy. \tag{1.31}$$

To simplify notation, let

$$P_0 f(x) = \int K(x-y)\, f(y)\, dy, \tag{1.32}$$

and call $K(x)$ the Littlewood–Paley kernel. By (1.29), for any M,

$$|K(x)| \lesssim_M \frac{1}{(1+|x|)^M}, \tag{1.33}$$

and for any $j \in \mathbf{Z}$, (1.22) implies

$$|K_j(x)| = 2^{jd}|K(2^j x)| \lesssim_M \frac{2^{jd}}{(1+2^j|x|)^M}. \tag{1.34}$$

Proof For $|t| \leq 2^{-2k}$, direct integration and the support of $\phi_k(\xi)$ implies

$$(2\pi)^{-d} \int e^{ix\cdot\xi} e^{-it|\xi|^2} \phi_k(\xi)\, d\xi \lesssim_d 2^{dk}.$$

For $|t| > 2^{-2k}$ and $|x| \leq 2^{k+4}|t|$, the same stationary phase argument that gives (1.19) implies

$$|K_k(t,x)| \lesssim \frac{1}{|t|^{d/2}}.$$

Next suppose $|x| > 2^{k+4}|t|$. Then

$$\int e^{ix\cdot\xi} e^{-it|\xi|^2} \phi_k(\xi)\, d\xi = \int \phi_k(\xi) \frac{(-ix+2it\xi)\cdot\nabla_\xi}{|x-2t\xi|^2} \left(e^{ix\cdot\xi - it|\xi|^2}\right),$$

so for any M,

$$(1.27) = \int \phi_k(\xi) \left(\frac{(-ix+2it\xi)\cdot\nabla_\xi}{|x-2t\xi|^2}\right)^M \left(e^{ix\cdot\xi-it|\xi|^2}\right) d\xi. \tag{1.35}$$

Integrating by parts, when $|x| > 2^{k+4}|t|$, $|x-2t\xi| \sim |x|$, so

$$(1.35) \lesssim_M \frac{2^{kd}|t|^M}{|x|^{2M}} + \frac{2^{kd}}{|x|^M 2^{kM}} \lesssim \frac{2^{kd}}{|x|^M 2^{kM}}.$$

This proves (1.29). Equation (1.30) also follows from (1.35) since $|x-2t\xi| \sim |t||\xi|$ when $|x| < 2^{k-4}|t|$. □

Now choose $\chi(x) \in C_0^\infty(\mathbf{R})$ such that χ is supported on $|x| \leq 2$ and

$$\sum_{m\in\mathbf{Z}} \chi(x-m) = 1 \tag{1.36}$$

for all $x \in \mathbf{R}$. If $u_0 \in L^1(\mathbf{R})$, then for any N,

$$|x|^N \chi(x) u_0(x) \in L^1(\mathbf{R}).$$

Therefore, by (1.4) and (1.5),

$$\hat{f}(\xi) = \chi(\xi) \mathscr{F}(\chi(x) u_0(x)) \tag{1.37}$$

is a Schwartz function. By the Sobolev embedding theorem, if f is given by (1.37),

$$\|e^{it\Delta}f\|_{L^\infty(\mathbf{R})} \lesssim \|\chi(x)u_0\|_{L^1(\mathbf{R})}.$$

Following the computations in the proof of Theorem 1.7,

$$\begin{aligned} e^{it\Delta}f &= (2\pi)^{-1}\int e^{ix\xi}e^{-it\xi^2}\chi(\xi)\left[\int e^{-iy\xi}\chi(y)u_0(y)\,dy\right]d\xi \\ &= (2\pi)^{-1}\int\left[\int \chi(\xi)e^{i(x-y)\xi}e^{-it\xi^2}d\xi\right]\chi(y)u_0(y)\,dy \\ &= (2\pi)^{-1}\int\left[\int e^{-it\left(\xi-\frac{x-y}{2t}\right)^2}\chi(\xi)\,d\xi\right]e^{i\frac{(x-y)^2}{4t}}\chi(y)u_0(y)\,dy. \end{aligned}$$

Plugging in $t=1$ and making stationary phase arguments,

$$e^{it\Delta}f|_{t=1} = \int \widetilde{K}(x-y)\chi(y)u_0(y)\,dy,$$

where for any M,

$$|\widetilde{K}(x-y)| \lesssim_M \frac{1}{(1+|x-y|)^M}. \tag{1.38}$$

Now utilize the Galilean transformation.

Lemma 1.8 (Galilean transformation) *If u solves the linear Schrödinger equation* (1.1)*, then for any $\xi_0 \in \mathbf{R}^d$,*

$$e^{-it|\xi_0|^2}e^{ix\cdot\xi_0}u(t,x-2t\xi_0) \tag{1.39}$$

solves (1.1) *with initial data $e^{ix\cdot\xi_0}u_0(x)$.*

For any $m \in \mathbf{Z}$, set

$$\hat{f}_m(\xi) = \chi(\xi-m)\mathscr{F}(\chi(x)u_0(x)),$$

and then by (1.38) and (1.39),

$$e^{it\Delta}f_m|_{t=1} = \int \widetilde{K}_m(x-y)\chi(y)u_0(y)\,dy,$$

where for any M,

$$|\widetilde{K}_m(x-y)| \lesssim_M \frac{1}{(1+|x-2tm-y|)^M}.$$

Therefore,

$$\|e^{it\Delta}(\chi u_0)|_{t=1}\|_{L^\infty(\mathbf{R})} \lesssim \|\chi(x)u_0(x)\|_{L^1(\mathbf{R})}. \tag{1.40}$$

Then by (1.36) and (1.40),

$$\| e^{it\Delta} u_0|_{t=1} \|_{L^\infty(\mathbf{R})} \lesssim \| u_0 \|_{L^1(\mathbf{R})}. \tag{1.41}$$

Taking a d-dimensional partition of unity and decomposing

$$\hat{u}_0(\xi) = \sum_{m \in \mathbf{Z}^d} \chi(\xi - m) \mathscr{F}(\chi(x) u_0(x))$$

proves the same in any dimension when $t = 1$.

Time reversal symmetry and the scaling symmetry generalize (1.41) to any t.

Lemma 1.9 (Scaling symmetry) *If u solves* (1.1), *then for any $\lambda > 0$,*

$$\lambda^{d/2} u\left(\lambda^2 t, \lambda x\right) \tag{1.42}$$

solves (1.1) *with initial data $\lambda^{d/2} u_0(\lambda x)$.*

Remark The invariance in (1.42) is called the mass-critical scaling.

Taking $\lambda = t^{1/2}$, (1.41) implies

$$\lambda^{d/2} \| u(\lambda^2, \lambda x) \|_{L^\infty(\mathbf{R}^d)} \lesssim \lambda^{d/2} \| u_0(\lambda x) \|_{L^1(\mathbf{R}^d)} = \lambda^{-d/2} \| u_0 \|_{L^1(\mathbf{R}^d)},$$

and therefore,

$$\| u(t,x) \|_{L^\infty(\mathbf{R}^d)} \lesssim t^{-\frac{d}{2}} \| u_0 \|_{L^1(\mathbf{R}^d)}. \tag{1.43}$$

1.2 Strichartz Estimates

The inhomogeneous version of (1.1) is given by

$$iu_t + \Delta u = F, \quad u(0,x) = 0. \tag{1.44}$$

By unitary group theory, the solution to (1.44) is formally given by

$$u(t) = -i \int_0^t e^{i(t-\tau)\Delta} F(\tau)\, d\tau.$$

The dispersive estimate (1.20) implies a space-time integrability estimate for a solution to (1.44).

Lemma 1.10 (Inhomogeneous estimate) *Suppose $d\left(\frac{1}{2} - \frac{1}{q}\right) < 1$; $\tilde{p}$, $p < \infty$, and*

$$d\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{1}{\tilde{p}} + \frac{1}{p}.$$

Then

$$\left\| \int_{-\infty}^t e^{i(t-\tau)\Delta} F(\tau)\, d\tau \right\|_{L_t^p L_x^q(\mathbf{R} \times \mathbf{R}^d)} \lesssim_{d,p,\tilde{p},q} \| F \|_{L_t^{\tilde{p}'} L_x^{q'}(\mathbf{R} \times \mathbf{R}^d)}. \tag{1.45}$$