Mathematics for Physicists

This textbook is a comprehensive introduction to the key disciplines of mathematics – linear algebra, calculus and geometry – needed in the undergraduate physics curriculum. Its leitmotiv is that success in learning these subjects depends on a good balance between theory and practice. Reflecting this belief, mathematical foundations are explained in pedagogical depth, and computational methods are introduced from a physicist's perspective and in a timely manner. This original approach presents concepts and methods as inseparable entities, facilitating in-depth understanding and making even advanced mathematics tangible.

The book guides the reader from high-school level to advanced subjects such as tensor algebra, complex functions and differential geometry. It contains numerous worked examples, info sections providing context, biographical boxes, several detailed case studies, over 300 problems and fully worked solutions for all odd-numbered problems. An online solutions manual for all even-numbered problems will be made available to instructors.

Alexander Altland is Professor of Theoretical Physics at the University of Cologne. His areas of specialization include quantum field theory and the physics of disordered and chaotic systems. He is co-author of the hugely successful textbook *Condensed Matter Field Theory* (2nd edition published by Cambridge University Press, 2010). He received the Albertus Magnus Teaching Award of the faculty of mathematics and natural sciences of Cologne University.

Jan von Delft is Professor of Theoretical Physics at the Arnold Sommerfeld Center for Theoretical Physics at the Ludwig–Maximilians–Universität in Munich. His research is focused on mesoscopic physics and strongly interacting electron systems. For his engagement in teaching, utilizing electronic chalk and "example+practice" problem sheets including example problems with detailed solutions, he received a Golden Sommerfeld teaching award.

Mathematics for Physicists

Introductory Concepts and Methods

ALEXANDER ALTLAND
Universität zu Köln

JAN VON DELFT
Ludwig–Maximilians–Universität München

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Preface

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.
>
> Eugene Paul Wigner

This text is an introduction to mathematics for beginner physics students. It contains all the material required in the undergraduate curriculum. The main feature distinguishing it from the large number of available books on the subject is that mathematical *concepts* and *methods* are presented in unison and on an equal footing. Let us explain what is meant by this statement.

Physicists teaching mathematics often focus on the *training of methods*. They provide recipes for the algebraic manipulation of vectors and matrices, the differentiation of functions, the computation of integrals, etc. Such pragmatic approaches are often justified by time pressure: physics courses require advanced mathematical methodology and students have to learn it as quickly as possible.

However, knowledge of computational methods alone will not carry a student through the physics curriculum. Equally important, she needs to understand the mathematical principles and *concepts* behind the machinery. For example, the methodological knowledge that the derivative of x^2 equals $2x$ remains hollow, unless the conceptual meaning of that $2x$ as a local linear approximation of a parabola is fully appreciated. Similar things can be said about any of the advanced mathematical methods required in academic physics teaching.

Recognizing this point, physics curricula often include lecture courses in pure mathematics – who would be better authorized to teach mathematical concepts than mathematicians themselves? However, there is a catch: mathematicians approach the conceptual framework of their science from a perspective different from that of physicists. Rigorous proofs and existence theorems stand in the foreground and are more important than the communication of concepts relevant to the understanding of structures in physics. This tendency is pervasive, including when mathematicians teach "mathematics for physicists".

For these reasons, the traditional division – physics courses focusing on methods, mathematics courses on proofs – is not ideal.

Pedagogical strategy – unified presentation of concepts and methods

This book aims to bridge the divide. It contains a *unified presentation* of concepts and methods, written from the perspective of theoretical physicists. Mathematical structures are motivated and introduced as an orienting framework for the understanding of methods. Although less emphasis is put on formal proofs, the text maintains a fair level of mathematical hygiene and generally does present material in a formally consistent manner. Importantly, it does not operate on a higher level of technicality or abstraction than is standard in physics.

As an example, consider *vectors*. First-time introductions often focus on three-dimensional vectors, visualized as arrows and described by three components. To many students this picture is familiar from high school, and it suffices to follow introductory mechanics courses in university. However, only one year later quantum mechanics is on the agenda. The mathematics of quantum mechanics is all about vectors, however these now live in a more abstract (Hilbert) space which is hard to visualize. This can be frustratingly difficult for students conditioned to a narrow understanding of vectors. In this text, we take the different approach of introducing vector spaces in full generality at a very early stage. Experience shows that beginner students have no difficulty in absorbing the concept. Once familiar with it, the categorization even of very abstract objects as vectors feels natural and does not present any difficulty. In this way, the later mathematics of quantum mechanics becomes much easier to comprehend.

Does the enhanced emphasis on concepts come at the expense of methodological training? The answer is an emphatic “no!” – a solid conceptual understanding of mathematics leads to greatly improved practical and methodological skills. These statements are backed by experience. The book is based on a course taught more than ten times to first-year students at the University of Cologne and Ludwig–Maximilians–Universität (LMU) Munich. Building on this text, these courses introduce mathematical methods at a pace compatible with standard physics curricula and at load levels manageable for average students. The introduction of this new teaching concept has significantly enhanced the students’ performance and confidence. Its emphasis on the motivation of mathematical concepts also provides welcome tail wind in the understanding of concurrent courses in pure mathematics.

Organization and scope

The book is organized into *three parts*:

- ▹ Linear Algebra (L),
- ▹ Calculus (C),
- ▹ Vector Calculus (V).

Starting at high-school level, each part covers the material required in a standard Bachelor curriculum and reaches out somewhat beyond that. In fact, the whole text has been written with an eye on modern developments in physics research. This becomes apparent in the final chapters which include introductions to multilinear algebra, complex calculus, and differential forms, formulated in the language used in contemporary research. However, the early chapters are already formulated in ways which anticipate these developments and occasionally employ language and notation slightly different from (but never incompatible with) that of traditional teaching. Generally, the writing style of each part gradually changes from moderately paced and colloquial at the beginning to somewhat more concise and “scientific” towards the final chapters. Due to its modular structure, the text can also serve as a reference covering all elements of linear algebra, calculus and vector calculus encountered in a Bachelor physics curriculum.

The reading order of parts L, C, V is not fixed and can be varied according to individual taste and/or time constraints. A good way to start is to first read a few chapters of each of parts L and C and then move into V. Where later chapters draw connections between fields, initial *remarks* state the required background so that there is no risk of accidentally missing out on something essential. For concreteness, Table 1 details the organization of a one-semester course at LMU Münich. Table 2 is the outline of a more in-depth two-semester course at Cologne University where the first and second semesters focus on calculus and linear algebra, respectively.

Pedagogical features

Many beginning physics students struggle with mathematics. When confronted with abstract material they ask the "what for?" question or even perceive mathematics as a hostile subject. By contrast, the authors of the present text love mathematics and know that the symbiotic relationship between the disciplines is a gift. They have tried to convey as much as possible of this positive attitude in the text.

Examples, info sections, case studies, biographical boxes

The text includes numerous *examples* showing the application of general concepts or methods in physically motivated contexts. It also contains more than a hundred *info sections* addressing the background or relevance of mathematical material in physics. For example, the info sections on pp. 52 and 113 put general material of linear algebra into the context of Einstein's theory of relativity. A few *case studies*, more expansive in scope than the info sections, illustrate how mathematical concepts find applications in physics. For example, quantum mechanics is mentioned repeatedly throughout Part L. All these references are put into context in a case study (Section L8.4) discussing how the principles of quantum mechanics are naturally articulated in the language of linear algebra.

Almost all info sections and case studies can be read without further background in physics. However, it should be emphasized that this text is not an introduction to physics and that the added material only serves illustrative purposes. It puts mathematical material into a physics context but remains optional and can be skipped if time pressure is high and priorities have to be set.

Finally, abstract mathematical material often feels less alien if the actual person responsible for its creation is visible. Therefore numerous *biographical boxes* portray some of the great minds behind mathematical or physical invention.

Problems

Solving *problems* is an essential part of learning mathematics. About one third of the book is devoted to problems, more than 300 in number, all tried and tested in Munich and Cologne. In this text, there is an important distinction between odd- and even-numbered problems: the odd-numbered *example problems* include detailed solutions serving as efficient and streamlined templates for the handling of a technical task. They can be used for self-study or for discussion in tutorials. These exemplar problems prepare the reader for the subsequent even-numbered *practice problems*, which are of similar structure but should be solved independently.

To avoid disruption of the text flow, all problems are assembled in three separate chapters, one at the end of each part. Individual problems are referenced from the text location where they first become relevant. For example, → L5.5.1-2, referenced in Section L5.5 on *general linear maps and matrices*, points to an example problem on two-dimensional rotation matrices, followed by a practice problem on three-dimensional ones. Three chapters at the very end of the book contain the solutions to the odd-numbered problems. A password-protected manual containing solutions of all even-numbered problems will be made available to instructors.

Index, margin, hyperlinks and English language

Keywords appearing in the *index* are highlighted in **bold** throughout the text. Likewise in bold, we have added a large number of *margin keywords*. Margin keywords often duplicate index keywords for extra visibility. More generally they represent topical name tags providing an at-a-glance overview of what is going on on a page. *Slanted font* in the text is used for emphasis or to indicate topical structure.

The electronic version of this book is extensively hyperlinked. Clicking on a page number cited in the text causes a jump to that page, and similarly for citations of equations, chapters, sections, problems and index keywords. Likewise, clicking on the title of an odd-numbered problem jumps to its solution, and vice versa.

Finally, a word of encouragement for readers whose mother tongue isn't English: Learning to communicate in English – the lingua franca of science – at the earliest possible stage is more important then ever. This is why we have written this text in English and not in our own native language. Beginners will find that technical texts like this one are much easier to read than prose and that learning scientific English is easier than expected.

Some remarks for lecturers

We mentioned above that the present text deviates in some points from standard teaching in physics. None of these changes are drastic, and most amount to a slightly different accentuation of material. We already mentioned that we put emphasis on the general understanding of vectors. Students conditioned to "seeing vectors everywhere" have no difficulties in understanding the concept of spherical harmonics as a complete set of functions on the sphere, interpreting the Fourier transform as a basis change in function space, or thinking of a Green function as the inverse of a linear operator. We know from experience that once this way of thinking has become second nature the mathematics of quantum mechanics and of other advanced disciplines becomes much easier to comprehend.

On a related note, the physics community has the habit of regarding every object comprising components as either a vector or a matrix. However, only a fraction of the index-carrying objects encountered in physics are genuine vectors or matrices.[1] Equally important are dual vectors, bilinear forms, alternating forms, or tensors. Depending on the field one is working in, the "everything-is-a-vector" attitude can be tolerable or a notorious

[1] For example, a magnetic field "vector" does not change sign under a reflection of space. It therefore cannot be a true vector, which always causes confusion in teaching.

source of confusion. The latter is the case in fields such as particle physics and relativity, and in emerging areas such as quantum information or topological condensed matter physics. Linear algebra as introduced in this text naturally accommodates non-vectorial and non-matrix objects, first examples including the cross product of vectors and the metric of vector spaces. In the later parts of the text we introduce tensors and differential forms, and illustrate the potency of these concepts in a case study on electromagnetism (Chapter V7).

One of the less conventional aspects of this text is the use of *covariant notation* (indices of vector components upstairs, those of vectors downstairs). Covariant notation has numerous pedagogical advantages, both pragmatic and conceptual. For instance, it is very efficient as an error tracking device. Consistent summations extend over pairs of contravariant superscript and covariant subscript indices, and violations of this rule either indicate an error (a useful consistency check) or the hidden presence of "non-vectorial structures" (the latter occurring in connection with, e.g., the cross product). In all such cases, we explain what is going on either right away, or somewhat later in the text. In our teaching experience, the covariant approach is generally well received by students. As added value, it naturally prepares them for fields such as relativity or particle physics, where it is mainstream. (Readers consulting this text as a secondary reference and for which covariance does not feel natural are free to ignore it – just read all indices in a traditional way as subscripts.)

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Alexander Altland and Jan von Delft

Table 1 Outline of a moderately paced (top) or fast-paced (bottom) one-semester course based on this text. Each row refers to a 90-minute lecture.

	L	C	V	Topic
1	1.1-2			Basic concepts I: sets, maps and groups
2	1.3			Basic concepts II: fields and complex numbers
3		1		Differentiation of one-dimensional functions
4		2		Integration of one-dimensional functions
5	2.1-3			Vector spaces: standard vector space, general definition
6	2.4-5			Vector spaces: basis and dimension
7	3.1-2			Euclidean spaces I: scalar product, norm, orthogonality
8	3.3-4			Euclidean spaces II: metric, complex inner product
9	4			Vector product: Levi–Civita symbol, various identities
10			1	Curves, line integrals
11		3		Partial differentiation
12		4.1		Multidimensional integration I: Cartesian
13			2.1-3	Curvilinear coordinates: polar, cylindrical, spherical
14		4.2-3		Multidimensional integration II: curvilinear coordinates
15			3.1-2	Scalar fields and gradient
16			3.4	Vector fields: gradient fields, nabla operator
17	5.1-3			Linear maps I: matrices, matrix multiplication
18	5.4-6			Linear maps II: inverse, basis transformations
19	6		2.5	Determinants: definition, properties
20	7			Diagonalization: eigenvalues, eigenvectors
21		5.1-2		Taylor series: definition, complex Taylor series
22		7.1-3		Differential equations I: separable DEQs, linear first-order DEQs
23		7.4-5		Differential equations II: systems of linear DEQs
24		6.1-2		Fourier calculus I: Dirac delta function, Fourier series
25		6.3		Fourier calculus II: Fourier transforms
26		4.4-5	3.5	Integration in arbitrary dimensions; flux integrals
27			3.5	Sources of vector fields, Gauss's theorem
28			3.6-7	Circulation of vector fields, Stokes's theorem
1	1			Basic concepts I: sets, maps, groups, fields and complex numbers
2		1,2		Differentiation and integration of one-dimensional functions
3	2			Vector spaces: definition, examples, basis and dimension
4	3			Euclidean spaces: scalar product, norm, orthogonality, metric
5	4			Vector product: Levi–Civita symbol, various identities
6			1	Curves, line integrals
7		3, 4.1		Partial differentiation, multidimensional integration: Cartesian
8			2.1-3	Curvilinear coordinates: polar, cylindrical, spherical
9		4.2-4		Multidimensionl integration, curvilinear coordinates
10			3.1-2	Scalar fields and gradient
11			3.3-4	Extrema of functions with constraints, gradient fields
12	5.1-3			Linear maps I: matrices, matrix multiplication
13	5.4-6			Linear maps II: inverse, basis transformations
14	6	4.5	2.5	Determinants: definition, properties, applications
15	7			Diagonalization: eigenvalues, eigenvectors
16	8			Orthogonal, unitary, symmetric and Hermitian matrices
17		5.1-2		Taylor series: definition, complex Taylor series
18		7.1-3		Differential equations I: separable DEQs, linear first-order DEQs
19		7.4-5		Differential equations II: systems of linear DEQs
20		5.3-5		Perturbation expansions; higher-dimensional Taylor series
21		6.1-2		Fourier calculus I: Dirac delta function, Fourier series
22		6.3		Fourier calculus II: Fourier transforms
23		6.3,7.5		Fourier series for periodic functions; Green functions
24		7.6-7		Differential equations III: general nth-order DEQs, linearization
25			3.5	Sources of vector fields, Gauss's theorem
26			3.6-7	Circulation of vector fields, Stokes's theorem
27		9.1-2		Holomorphic functions, complex integration, Cauchy's theorem
28		9.3-5		Singularities, residue theorem, essential singularities

Table 2 Outline of a more in-depth two-semester course based on this text.

	L	C	V	Topic
1	1.1-2			Basic concepts I: sets, maps and groups
2	1.3-4			Basic concepts II: fields and complex numbers
3	2.1-3			Vector spaces I: standard vector space and general definition
4	2.4-5			Vector spaces II: basis and dimension
5	3			Euclidean geometry: scalar product, norm, orthogonality
6	4			Vector product
7		1		Differentiation of one-dimensional functions
8		2		Integration of one-dimensional functions
9		3		Partial differentiation
10		4.1		Multidimensional integration in Cartesian coordinates
11			1.1-2	Curves
12			1.3-4	Curve length and line integrals
13			2.1-2	Curvilinear coordinates I: polar coordinates, general concept
14			2.3-4	Curvilinear coordinates II: cylindrical, spherical
15			3.1-2	Scalar fields and gradient
16			3.4	Gradient fields
17		4.2-3		Curvilinear integration in two and three dimensions
18		4.4		Curvilinear surface integrals
19		5.1-2		Taylor series: definition, complex Taylor series
20		6.1		Fourier calculus I: Dirac delta function
21		6.2		Fourier calculus II: Fourier series
22		6.3		Fourier calculus III: Fourier transform
23		6.3-4		Fourier transform applications
24			3.5	Flux integrals of vector fields
25			3.5	Sources of vector fields, Gauss's theorem
26			3.6-7	Circulation of vector fields, Stokes's theorem
1	5.1-2			Linear maps and matrices
2	5.3-4			Matrix multiplication, and inverse
3	5.4			Dimension formula, linear systems of equations
4	5.5-6			Basis transformations I
5	5.6			Basis transformations II
6	6.1			Determinants I
7	6.1			Determinants II
8		4.2-5	2.5	Integration in arbitrary dimensions, revisited
9	7.1-3			Eigenvalues, eigenvectors, characteristic polynomial
10	7.4-5			Diagonalization of matrices
11	7.5	7.1		Differential equations (DEQ): motivation
12		7.2-3		DEQs: typology and linear first-order equations
13		7.3-4		Systems of first-order DEQs
14		7.4		Green functions
15		7.5		General first-order DEQs
16		7.6		*n*th-order differential equations
17		7.7-9		Linearization, fixed points, partial differential equations
18	8.1-2			Linear maps: unitary and orthogonality
19	8.3			Linear maps: Hermiticity and symmetry
20	8.4			Case study: linear algebra in quantum mechanics
21		9.1		Holomorphic functions
22		9.2		Complex integration, Cauchy's theorem
23		9.3-4		Singularities, residue theorem
24	9.1-2			Linear algebra in function spaces I
25	9.3-4			Linear algebra in function spaces II