L

LINEAR ALGEBRA

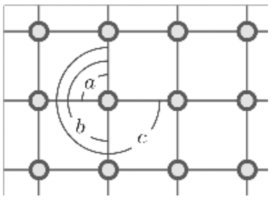
The first part of this book is an introduction to linear algebra, the mathematical discipline of structures that are, in a sense to be discussed, “straight”. No previous knowledge of the subject is assumed. We start with an introduction to various basic structures in mathematics: sets, groups, fields, different types of “numbers”, and finally vectors. This is followed by a discussion of elementary geometric operations involving vectors, the computation of lengths, angles, areas, volumes, etc. We then explain how to describe relations between vectorial objects via so-called linear maps, how to represent linear maps in terms of matrices, and how to work with these operations in practice. Part L concludes with two chapters on advanced material. The first introduces the interpretation of functions as vectors (a view of essential importance to quantum mechanics). In the second, we discuss linear algebra in vector spaces containing a high level of intrinsic structure, so-called tensor spaces, which appear in disciplines such as relativity theory, fluid mechanics and quantum information theory.

L1 Mathematics before numbers

Many people believe that numbers are the most basic elements of mathematics. This, however, is an outside view which does not reflect the way mathematics itself treats numbers. Numbers can be added, subtracted, multiplied and divided by, which means that they possess a considerable degree of complexity.[1] Metaphorically speaking, they are high up in the evolutionary tree of mathematics, and beneath them there exist numerous structures of lesser complexity. Much as a basic understanding of evolutionary heritage is important in understanding life – reptiles, vs. birds, vs. mammals, etc. – the evolutionary ancestry of numbers is a key element in the understanding of mathematics, and physics. We take this as motivation to start with a synopsis of various pre-numerical structures which we will later see play a fundamental role throughout the text.

symmetry operations

EXAMPLE Consider a two-dimensional square lattice that is invariant under rotations by 90 degrees (i.e. if you rotate the lattice by 90 deg[2] it looks the same as before, see figure). Then rotations by 0, 90, 180 or 270 deg are "*symmetry operations*" that map the lattice onto itself. Let us denote these operations by e, a, b and c, respectively. Two successive rotations by 90 deg are equivalent to one by 180 deg, a fact we may express as $a \cdot a = b$. Similarly, $b \cdot b = e$ (viewing a 360 deg rotation as equivalent to one by 0 deg). These operations are examples of mathematical objects which can be "combined" with each other, but not "divided" by one another. Together, they form a pre-number structure, soon to be identified as a "group". Generically groups have less structure than numbers and yet are very important in physics.



L1.1 Sets and maps

When we work with a complex systems of objects of *any* kind we need ways to categorize and store them. At the very least, we require containers capable of storing objects (think of the situation in a repair shop). On top of that one may want to establish connections between the objects of different containers (such as a tabular list indicating which screw

[1] At the end of the nineteenth century mathematicians became increasingly aware of gaps in the logical foundations of their science. It became understood that the self-consistent definition even of natural numbers $(1, 2, 3, \ldots)$ was more complex than was previously thought. For an excellent account of the ensuing crisis of mathematics, including its social dimensions, we refer to the graphic novel *Logicomix*, A. Doxiadis, Bloomsbury Publishing, 2009.

[2] In this text we use the standard abbreviation "deg" for degrees.

in the screw-box matches which screwdriver in the screwdriver rack). In the terminology of mathematics, containers are called "sets", and the connections between them are established by "maps". In this section we define these two fundamental structures and introduce various concepts pertaining to them.

Sets

set

Perhaps the most basic mathematical structure is that of a **set**. (The question whether there are categories even more fundamental than sets is in fact a subject of current research.) As indicated above, one may think of a set as a container holding objects. In mathematical terminology, the objects contained in a set are called its **elements**. Unlike the containers in a repair shop, mathematical sets are not "physical" but simply serve to group objects according to certain categories (which implies that one object may be an element of different sets). For example, consider the set of all your relatives. Your mother is an element of that set, and at the same time one of the much larger set of all females on the planet, etc. More formally, the notation $a \in A$ indicates that a is an element of the set A, and $A = \{a, b, c, \ldots\}$ denotes the full set.

notation

INFO Be careful to exercise *precision in matters of notation.* For example, denoting a set by $(a, b, c, \ldots)$ would be incompatible with the standard curly bracket format $\{a, b, c, \ldots\}$ and an abuse of notation. Insistence on clean notation has nothing to do with pedantry and serves multiple important purposes. For example, the notation $B = \{1, 2, 3\}$ is understood by every mathematically educated person on the planet, meaning that standardized mathematical notation makes for the most international idiom there is. At the same time, uncertainties in matters of notation often indicate a lack of understanding of a concept. For example, $a \in \{a\}$ is correct notation indicating that a is an element of the set $\{a\}$ containing just this one element. However, it would be incorrect to write $a = \{a\}$. The element a and the one-element set $\{a\}$ are different objects. Uncertainty in matters of notation is a sure and general indicator of a problem in one's understanding and should always be considered a warning sign – stop and rethink.

The definition of sets and elements motivates a number of generally useful secondary definitions:

empty set

- An **empty set** is a set containing no elements at all and denoted by $A = \{\}$, or $A = \emptyset$.
- A **subset** of A, denoted by $B \subset A$, contains some of the elements of A, for example, $\{a, b\} \subset \{a, b, c, d\}$. The notation $B \subseteq A$ indicates that the subset B may actually be equal to A. On the other hand, $B \subsetneq A$ means that this is certainly not the case.
- The **union** of two sets is denoted by $\cup$, for example, $\{a, b, c\} \cup \{c, d\} = \{a, b, c, d\}$. The **intersection** is denoted by $\cap$, for example, $\{a, b, c\} \cap \{c, d\} = \{c\}$.
- The removal of a subset $B \subset A$ from a set A results in the **difference**, denoted by $A \backslash B$. For example, $\{a, b, c, d\} \backslash \{c\} = \{a, b, d\}$.
- We will often define sets by *conditional rules*. The standard notation for this is set $=$ $\{\text{elements} \,|\, \text{rule}\}$. For example, with $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ the set of all even integers up to 10 could be defined as $B = \{a \in A \,|\, a/2 \in A\} = \{2, 4, 6, 8, 10\}$.

Cartesian product

▹ Given two sets A and B, the **Cartesian product**,[3]

$$A \times B \equiv \{(a,b) \,|\, a \in A, b \in B\}, \tag{L1}$$

is a set containing all pairs (a,b) formed by elements of A and B.

The number of elements of a set is called its **cardinality**. The cardinality can be finite (the set of all your relatives) or infinite (the set of all natural numbers). Among the infinite sets one distinguishes between "countable" and "uncountable" sets. A set is **countable** if one can come up with a way to number its elements. For example, the set of even integers $A = \{0, 2, 4, \dots\}$ is countable. The real numbers (see Section L1.3) form an uncountable set.

equivalence classes

It is often useful to organize sets in **equivalence classes** expressing the equality, $a \sim b$, of two elements relative to a certain criterion, R. For example, let A be the set of relatives and let the distinguishing criterion, R, be their sex. The notation Victoria $\sim$ Erna then indicates that the two relatives are equivalent in the sense that they are female. An equivalence relation has the following defining properties:

▹ reflexivity: $a \sim a$, every element is equivalent to itself,

▹ symmetry: $a \sim b$ implies $b \sim a$ and vice versa,

▹ transitivity: $a \sim b$ and $b \sim c$ implies $a \sim c$.

The subset of all elements equivalent to a given reference element a is called an *equivalence class* and denoted $[a] \subset A$. In the example of relatives and their sex, there are two such subsets, for example $A = [\text{Herbert}] \cup [\text{Erna}]$. The label used for an equivalence class is not unique; for example, one might relabel $[\text{Erna}] = [\text{Victoria}]$. The set of all equivalence classes relative to a criterion R is called its **quotient** and is denoted by A/R. In the example of relatives (A) and their sex (R), the quotient set $A/R = \{[\text{Herbert}], [\text{Victoria}]\}$ would have two elements, the class of males and that of females.

EXAMPLE Consider the set of integers, and pick some integer q. Now view any two integers as equivalent if they have the same *remainder under division by* q. For example, $q = 4$ defines $0 \sim 4 \sim 8$, $1 \sim 5 \sim 9$. In this case there are four equivalence classes, denotable by [0], [1], [2] and [3]. In general, the remainder of p divided by q is denoted by $p \bmod q$ (spoken "p-modulo-q", or just "p-mod-q"), e.g., $8 \bmod 4 = 0$, $6 \bmod 4 = 2$, or $-5 \bmod 4 = 3$ (by definition, remainders are taken to be positive). The equivalence class of all integers with the same remainder r under division by q is the set $[r] = \{p \in \mathbb{Z} | p \bmod q = r\}$. There are q such equivalence classes, and the set of these classes is denoted by $\mathbb{Z}_q \equiv \mathbb{Z}/q\mathbb{Z} = \{[0], [1], \dots, [q-1]\}$.

Maps

map

Consider two sets, A and B, plus a rule, F, assigning to each element a of A an element b of B. Such a rule, written as $F(a) \equiv b \in B$, is called a **map**. In mathematics and physics, maps are specified by the following standard notation:

[3] We follow a widespread convention whereby $\Box \equiv \triangle$ means "$\Box$ is defined by $\triangle$". In the German literature, the alternative notation $\Box := \triangle$ is frequently used.

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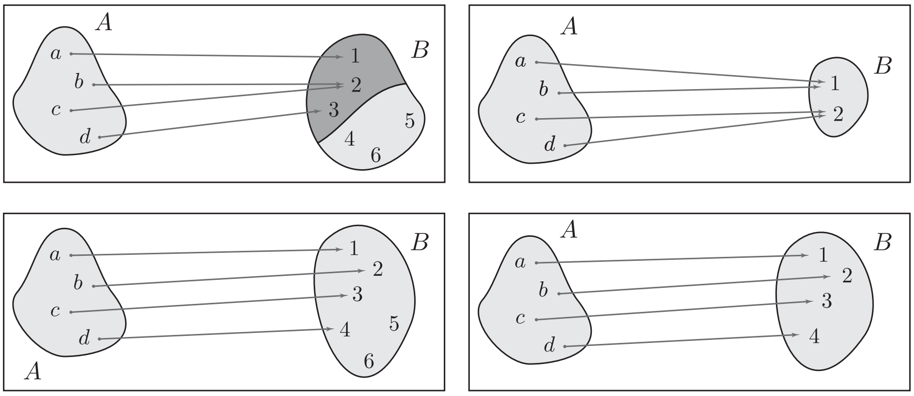


Fig. L1 Different types of maps. Top left: a generic map, top right: surjective map, bottom left: injective map, bottom right: bijective map.

$$F : A \to B, \qquad a \mapsto F(a). \tag{L2}$$

domain

The set A is called the **domain** of the map and B is its **codomain**.[4] An element $a \in A$ fed into the map is called an **argument** and $F(a)$ is its **value** or **image**. Note that different types of arrows are used for "domain $\to$ codomain" and "argument $\mapsto$ image".

image

The **image** of A under F, denoted by $F(A)$, is the set containing all image elements of F: $F(A) = \{F(a)|a \in A\} \subseteq B$ (see dark shaded area in the top left panel of Fig. L1). A map is called **surjective** (top right panel) if its image covers all of B, $F(A) = B$, i.e. if any element of the codomain is the image of at least one element of the domain. It is called **injective** (bottom left) if every element of the codomain is the image of at most one element of the domain. The map is **bijective** if it is both surjective and injective (bottom right panel), i.e. if every element $b \in B$ of the codomain is the image of precisely one element $a \in A$ of the domain. Bijective maps establish an unambiguous relation between the elements of the sets A and B. The one-to-one nature of this assignment means that it can be inverted: there exists an **inverse map**, $F^{-1} : B \to A$, such that $F^{-1}(F(a)) = a$ for every $a \in A$.

inverse map

composition of maps

Given two maps, $F : A \to B$ and $G : B \to C$, their **composition** is defined by substituting the image element of the first as an argument into the second:

$$G \circ F : A \to C, \qquad a \mapsto G(F(a)). \tag{L3}$$

For example, the above statement about bijective maps means that the composition of a bijective map F with its inverse, F^{-1}, yields the identity map: $F^{-1} \circ F : A \to A$, with $a \mapsto F^{-1}(F(a)) = a$.

Finally, a map F defined on a Cartesian product set, $A \times B$, is denoted as

$$F : A \times B \to C, \qquad (a, b) \mapsto c = F(a, b).$$

This map assigns to every pair (a, b) an element of C. For example, the shape of a sand dune can be described by a map, $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto h(x, y)$, where for each

[4] The designation "codomain" is standard in mathematics, but not in physics. Oddly, physics does not seem to have an established designation for the "target set" of a map.

point (x, y) in the plane, the function $h(x, y)$ gives the height of the dune above that point. (→ L1.1.1-2)

L1.2 Groups

Sets as such are just passive containers storing elements. Often, however, the elements of a set are introduced with the purpose of doing something with them. As an example, consider the set of 90 deg rotations, $R \equiv \{e, a, b, c\}$, introduced on p. 3. A two-fold rotation by 180 deg is equivalent to a non-rotation and this fact may be described as $b \cdot b = e$. Or we may say that $a \cdot b = c$, meaning that a 90 degree rotation following one by 180 degrees equals one by 270 degrees, etc. In this section, we define groups as the simplest category of sets endowed with an "active" operation on their elements.

Definition of groups

group

The minimal structure[5] which brings a set to life in terms of operations between its elements is called a **group**. Let A be a set and consider an operation, "$\cdot$", equivalently called a **group law** or **composition rule**, assigning to every pair of elements a and b in A another element, $a \cdot b$:

$$\cdot : A \times A \to A, \qquad (a, b) \mapsto a \cdot b. \tag{L4}$$

group axioms

This map defines a group operation provided that the following four **group axioms** are satisfied:[6]

(i) **Closure**: for all a and b in A the result of the operation $a \cdot b$ is again in A. (Although this condition is already implied by the definition (L4), it is generally counted as one of the group axioms.)

(ii) **Associativity**: for all a, b and c in A we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(iii) **Neutral element**: there exists an element e in A such that for every a in A, the equation $e \cdot a = a \cdot e = a$ holds.

Depending on context, the neutral element is also called the **identity element** or **null element**.

(iv) **Inverse element**: For each a in A there exists an element b in A such that $a \cdot b = b \cdot a = e$.

[5] This statement is not fully accurate. There is a structure even more basic than a group, the **semigroup**. A semigroup need not have a neutral element, nor inverse elements to each element. In physics, semigroups play a less prominent role than groups, hence we will not discuss them further.

[6] Mathematicians often formulate statements of this type in a more compact notation. Frequently used symbols include $\forall$, abbreviating **for all**, and $\exists$, for **there exists**. Expressed in terms of these, the group axioms read: (i) $\forall a, b \in A$, $a \cdot b \in A$. (ii) $\forall a, b, c \in A$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. (iii) $\exists e \in A$ such that $\forall a \in A$, $a \cdot e = e \cdot a = a$. (iv) $\forall a \in A, \exists b \in A$ such that $a \cdot b = b \cdot a = e$. Although this notation is less frequently used in physics texts, it is very convenient and we will use it at times.

Under these conditions, A and "$\cdot$" define a group as $G \equiv (A, \cdot)$. A group should always be considered a "double", comprising a set and an operation. It is important to treat the operation as an integral part of the group definition: there are numerous examples of sets, A, which admit two different group operations, "$\cdot$" and "$*$". The doubles $G = (A, \cdot)$ and $G' = (A, *)$ are then different groups. We finally note that in some cases it can be more natural to denote the group operation by a different symbol, "$+$", or "$*$", or "$\circ$", ...

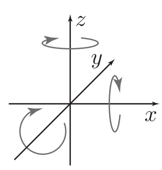
Nils Henrik Abel 1802–1829
Norwegian mathematician who made breakthrough contributions to several fields of mathematics before dying at a young age. Abel is considered the inventor (together with but independently from Galois) of group theory. He also worked on various types of special functions, and on the solution theory of algebraic equations.

EXAMPLE Here are a few first examples of groups.

- ▹ The simplest group of all, $G = (\{e\}, \cdot)$, contains just one element, its neutral element. Nothing much to discuss.
- ▹ The introductory example of 90 deg rotations, $R \equiv \{e, a, b, c\}$, defines a group of cardinality four. Its neutral element is e and for each element we have an inverse, for example $a \cdot c = e$. (Set up a "multiplication table" specifying the group operation for all elements of $R \times R$.) The same group, i.e. a set of four elements with the same group law, can be realized in different contexts. For example, for the quotient set $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$ defined on p. 5, a group operation may be defined as "addition modulo 4". This means that the addition of a number with remainder 1 mod 4 to one with remainder 3 mod 4 yields a number with remainder 0 mod 4, for example $[1] + [3] = [0]$. Set up the full group operation table for this group and show that it is identical to that of the group of 90 deg rotations discussed previously. This implies that $(\mathbb{Z}_4, +)$ and $(R, \cdot)$ define the same group. Explain in intuitive terms why this is so. The concept of different realizations of the same group is very important in both physics and mathematics. We will see many more examples of such correspondences throughout the text. (→ L1.2.2)
- ▹ **group $\mathbb{Z}_2$** The simplest nontrivial group, which nevertheless has many important applications, contains just two elements, $\mathbb{Z}_2 = \{e, a\}$, with $a \cdot a = e$. The **group** $\mathbb{Z}_2$ can be realized by rotations by 180 deg, as the group of integers with addition mod 2 (→ L1.2.1), and in many different ways. It plays a very important role in modern physics. For example, in information science, $\mathbb{Z}_2$ is the mathematical structure used to describe "**bits**", objects that can assume only one of two values, "on" and "off", or "0" and "1".
- ▹ The integers, $\mathbb{Z} \equiv \{\dots, -2, -1, 0, 1, 2, \dots\}$, with group operation "$+$" = "addition" (e.g. $2 + 4 = 6$) are an example of a group of infinite cardinality. $(\mathbb{Z}, +)$ has neutral element 0 and the inverse of a is $-a$, i.e. $a + (-a) = 0$. Why are the integers $(\mathbb{Z}, \cdot)$ with multiplicative composition $(2 \cdot 3 = 6)$ *not* a group?
- ▹ Another important example of a discrete group is the translation group on a lattice. (→ L1.2.3-4)

abelian group

If the group operation is *commutative* in the sense that it satisfies $a \cdot b = b \cdot a$ for all elements the group is called an **abelian group**. All examples mentioned so far have this property. **Non-abelian groups** possess at least some elements for which $a \cdot b \neq b \cdot a$. An important example is the group formed by all *rotations of three-dimensional space*. This group can be given a concrete realization by fixing three perpendicular coordinate axes in space. Any rotation can then be represented (see figure) as a succession of rotations around the



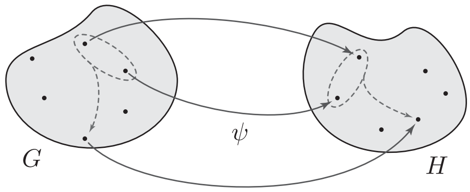


Fig. L2 The concept of a group homomorphism: a map between two groups that is compatible with the group operations (dashed) in that the image of the composition of two elements in the domain group (left) equals the composition of the image elements in the target group (right).

coordinate axes and the set of all these rotations forms a group where the group operation is the successive application of rotations. For example, $R_2 \cdot R_1$ is the rotation obtained by performing first R_1, then R_2. This concatenation is not commutative. For example, a rotation first around the x-axis and then around the z-axis is different from the operation in reverse order.

role in physics

INFO Groups play an important *role in physics*. This is because many classes of physical operations effectively carry a group structure. Simple examples include *rotations* or *translations* in space or time. These operations define groups because they can be applied in succession ("composed"), are associative, possess a neutral element (nothing is done), and can be inverted (undone). The translation and rotation groups play crucial roles in the description of momentum and angular momentum, respectively, both in classical and quantum mechanics. While continuous translations and rotations define groups of infinite cardinality, the physics of crystalline structures is frequently described in terms of finite restrictions. We mentioned the group $\mathbb{Z}_4$ of rotations by 90 deg around one axis as an example. In the late 1960s, group theory became important as a cornerstone of the *standard model* describing the fundamental structure of matter in terms of quarks and other elementary particles.

Despite the deceptive simplicity of the group axioms, the theory of groups is of great depth and beauty, and it remains a field of active research in modern mathematics.

Group homomorphism

group homomorphism

Above, we have seen that the same group structure can be "realized" in different ways. For example, the group $\mathbb{Z}_2$ can be realized as the group of rotations by 180 deg, or as addition in $\mathbb{Z}$ mod 2. Identifications of this type frequently appear in physics and mathematics, and it is worthwhile to formulate them in a precise language. To this end, consider two groups, $(G, \cdot)$ and $(H, \bullet)$ with *a priori* independent group operations. Let $\psi : G \to H$ be a map from G to H. If this map is such that for all $a, b \in G$ the equality $\psi(a \cdot b) = \psi(a) \bullet \psi(b)$ holds, then ψ is called a **group homomorphism** (see Fig. L2). The defining feature of a group homomorphism is its compatibility with the group law. As an example consider $G = H = (\mathbb{Z}, +)$, the addition of integers. Now assign to each integer its doubled value, $n \mapsto \psi(n) = 2n$. This map is a group homomorphism because $\psi(n+m) = 2(n+m) = 2n + 2m = \psi(n) + \psi(m)$. However, the map ϕ assigning to each integer its square, $n \mapsto \phi(n) = n^2$, is not a group homomorphism, because $\phi(1) + \phi(2) = 1^2 + 2^2 \neq \phi(1+2) = \phi(3) = 3^2$. As another example, consider the map $\psi : \mathbb{Z} \to \mathbb{Z}_2$, $n \mapsto \psi(n) = n \bmod 2$,

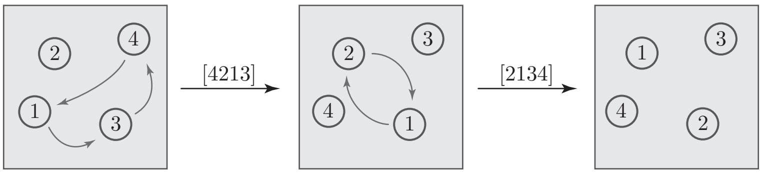


Fig. L3 Two permutations of four objects performed in succession.

assigning the number 0 or 1 to the integers, depending on whether n is even or odd. This is a homomorphism between the infinite group $(\mathbb{Z}, +)$ and the two-element group $\mathbb{Z}_2$.

A perfect identification between two groups G and H is obtained if there exists a *bijective* homomorphism between the two, a so-called **group isomorphism**. In this case, we write $G \cong H$. Mathematicians tend to not even distinguish between isomorphic groups, a view that can be confusing to physicists. The identification $\mathbb{Z}_2 \cong (\mathbb{Z} \bmod 2) \cong$ (rotations group by 180 deg) discussed above is a group isomorphism.

group isomorphism

EXERCISE Consider $\mathbb{Z}_n \equiv (\mathbb{Z} \bmod n, +)$, $n \in \mathbb{Z}$. Show that it defines a group of cardinality n. Show that $\mathbb{Z}_n$ is isomorphic to the group of rotations by $360/n$ deg around a fixed axis. (→ L1.2.2)

Permutation group

The permutations of n objects define one of the most important finite groups, the permutation group, S_n. Consider n arbitrary but distinguishable objects. For definiteness it may be useful to think of a set of n numbered billiard balls (see Fig. L3 for $n = 4$). A **permutation** is a rearrangement of these objects into a different order. For example, the reordering of four objects indicated in the left panel of the figure leads to the new arrangement shown in the middle. There are ***n*-factorial**, $n! \equiv n(n-1)(n-2)\ldots 1$, different arrangements or permutations,[7] and we consider the set, S_n, of cardinality $n!$ containing all of them.

permutation

factorial

Rearrangements can be composed, i.e. performed in succession. For example, the exchange in the middle panel of the figure leads to the final arrangement shown in the right panel. The group operation in S_n is this composition of permutations. Evidently, there is a trivial permutation (the one that leaves sequences unaltered), the composition of permutations is associative, and each permutation can be undone, such that there exists an inverse. This shows that S_n forms a group, the **permutation group** or **symmetric group** of n objects. It is easy to verify that the permutation group is non-abelian. (Invent examples of permutations proving the point.)

permutation group

Although the permutation group is easily defined, its mathematical structure is rather rich. (For example, the solution of a Rubik's cube amounts to a permutation of the 54 differently colored squares covering the six faces of the cube, and the solution algorithms

[7] One way to understand this number is to notice that the first of n objects can be put in any of n places. This leaves $n-1$ options for the second object (one position is already occupied by the first object), $n-2$ for the third, etc. The total number of rearrangements is obtained as the product of the number of options for objects $1, 2, \ldots, n$, i.e. $n(n-1)(n-2)\ldots 1 = n!$.