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To Kate and Amelia, with love

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Preface

Mathematicians are used to the notion of a subgroup of a group G as a subset containing the identity that is closed under taking products and inverses. However, it turns out that there are also circumstances in which we encounter subsets that are merely 'approximately closed'. Such sets arise, for example, in the construction of *expander graphs*, which are important in theoretical computer science, or in the study of *polynomial growth* in geometric group theory, which in turn has links to random walks and differential geometry. There are also numerous other examples.

A priori, there are a number of different ways of defining approximate closure. The notion that tends to arise in applications is that of *small doubling*, which we introduce in Definition 1.1.1. A more sophisticated and in some ways more tractable notion is the one that gives its name to this book and to the theory: that of an *approximate subgroup*. We introduce this in Definition 1.1.2. As we shall see, the two notions are intimately linked, and in some sense ultimately interchangeable. The aim of this book is to motivate and develop these notions in detail, with a view to leaving the reader in a position to understand and add to their growing literature, as well as to apply them elsewhere.

It turns out that the name *approximate subgroup* is justified by more than its origins as a notion of approximate closure. Indeed, we shall see in Sections 2.3 and 2.6 that many of the properties of approximate subgroups can be viewed as approximate versions of properties of 'exact' subgroups (as we will occasionally call genuine subgroups to emphasise their relationship to approximate subgroups). Understanding which properties of exact subgroups persist when we pass to approximate subgroups, and to what extent they persist, is an important part of the theory.

A striking feature of the theory of approximate groups is the range of fields that it uses and can be applied to. In this book alone we make heavy use of tools and ideas from combinatorics, convex geometry, group theory, representation theory and harmonic analysis, as well as touching on notions from probability. There are also substantial results in the literature on approximate groups that rest on algebraic group theory and model theory, although in order to keep the book reasonably focused we do not present these arguments here, instead directing the reader to the relevant references. Moreover, in addition to the applications to expander graphs and polynomial growth mentioned above, approximate groups have been applied to sieve theory, additive combinatorics, differential geometry and random walks, to name a few.

The applications of approximate groups are too numerous and diverse to present comprehensively in this book. However, since the applicability of approximate groups is one of their great selling points, it would seem remiss not to include at least some of them. We shall therefore go into some detail on certain applications to polynomial growth, and to geometric group theory more generally, in Chapter 11. The choice to present these particular applications rather than any others reflects my own interests as much as anything. For a taste of some other applications the reader may care to read Green's survey article [33].

One final important feature of the theory of approximate groups is that it is to some extent still being worked out. The elementary theory seems to be rather settled, but essentially all of the deeper results appearing in this book still have room for improvement. The book should therefore be thought of as giving a snapshot of the current state of an active research topic, rather than being a definitive description of its final form. Indeed, one of the main aims of the book is to equip the next generation of researchers with the tools and techniques that will enable them to take the field further. I hope that in a few years' time I will have the opportunity to write a revised version incorporating improvements to the theory made by the readers of the present version.

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I thank Ben Green for having introduced me to both approximate groups and growth in groups. Parts of Chapters 2–4 and Chapter 11 are inspired by two Part III courses he gave whilst I was a student in Cambridge, and I benefited from having access to some unpublished notes of his on growth.

I thank Emmanuel Breuillard, Pierre de la Harpe, Clare Dennison, Ben Green, Barnabás Janzer, Kate Marshall, Stuart Martin, Terence Tao, Romain Tessera, Jaspal Thandi and Matthew Wales for various helpful comments, corrections and discussions, as well as the anonymous referees whose comments improved the papers [72, 73] on which Chapters 6 and 8 are based. In 2019 I gave a Cambridge Part III course based on parts of this book, and I am also grateful to everyone who attended that course (some of whom are named above) for their comments, discussion and enthusiasm.

I am especially grateful to Clare Dennison at Cambridge University Press for her considerable support throughout the writing and production process, and Richard Hutchinson, the book's copyeditor, for a careful and thorough reading of the manuscript that improved its readability and considerably reduced the number of errors.

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