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Remarks on recent advances concerning boundary effects and the vanishing viscosity limit of the Navier–Stokes equations

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Abstract

This contribution covers the topic of my talk at the 2016-17 Warwick-EPSRC Symposium: "PDEs and their applications". As such it contains some already classical material and some new observations. The main purpose is to compare several avatars of the Kato criterion for the convergence of a Navier–Stokes solution, to a regular solution of the Euler equations, with numerical or physical issues like the presence (or absence) of anomalous energy dissipation, the Kolmogorov $\frac{1}{3}$ law or the Onsager $C^{0,\frac{1}{3}}$ conjecture. Comparison with results obtained after September 2016 and an extended list of references have also been added.

1.1 Introduction and uniform estimates.

In this contribution I will describe the main topics of my talk at the 2016-17 Warwick-EPSRC Symposium: *PDEs in Fluid Mechanics* in September 2016. Most of these issues are the results of a long term collaboration with Edriss Titi. I will also comment on some more recent (after September 2016) results (also collaboration with Edriss Titi and several other coworkers). In the same way I am going to include (mostly with no details) some recent results of other researchers and an extended list of references whenever they contribute to the understanding of the problems. Eventually one of the guidelines is the comparison between the use of weak convergence and the use of a statistical theory of turbulence. Hence the paper is organized as follows. After introducing some basic and well-known estimates, the zero viscosity limit of solutions of the Navier–

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Stokes equations is considered with no-slip boundary condition but in the presence, for the same initial data, of a Lipschitz solution of the Euler equations. This leads to an extension of Kato's theorem and to the introduction of several (equivalent) criteria for convergence to a smooth solution and for the absence of anomalous energy dissipation. Comparison of these criteria with physical observations or classical ansatz are made. In particular emphasis is given to the issue of the anomalous energy dissipation which leads to the comparison with the Kolmogorov $\frac{1}{3}$ law in the statistical theory of turbulence. Then this leads to the issue of the Onsager $C^{0,\frac{1}{3}}$ conjecture.

As a starting point consider solutions of the Euler equations and of the Navier–Stokes equations in a space-time domain

$$\Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}_t^+, \quad d = 2, 3.$$

We assume that the boundary $\partial\Omega$ is a C^1 manifold with $\vec{n}(x)$ denoting the outward normal at any point x in $\partial\Omega$. Then we introduce the function

$$d(x) = d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y| \geq 0$$

and the set

$$\mathcal{U}_\eta = \{x \in \Omega\,, d(x) < \eta\},$$

which have the following classical geometrical properties.

Proposition 1.1 *For $0 < \eta < \eta_0$ small enough $d(x)_{|\mathcal{U}_\eta} \in C^1(\overline{\mathcal{U}_\eta})$ and for any $x \in \mathcal{U}_\eta$ there exists a unique point $\sigma(x) \in \partial\Omega$ such that $d(x) = |x - \sigma(x)|$. Moreover for every $x \in \mathcal{U}_\eta$ we have*

$$x = \sigma(x) - d(x)\vec{n}(\sigma(x)) \quad \text{and} \quad \nabla_x d(x) = -\vec{n}(\sigma(x)). \tag{1.1}$$

To focus on the boundary effects, first, we consider a smooth (Lipschitz) solution $u(x,t)$ of the incompressible Euler equations with the impermeability condition:

$$\begin{aligned} &\nabla \cdot u = 0 \quad \text{and} \quad \partial_t u + u \cdot \nabla u + \nabla p = 0 \text{ in } \Omega \times [0,T] \\ &\text{and} \quad u \cdot \vec{n} = 0 \quad \text{on } \partial\Omega \times [0,T]\,. \end{aligned} \tag{1.2}$$

The value of such solution for $t = 0$ is denoted by $u_0(x) = u(x,0)$. For the same initial data $u_\nu(x,0) = u_0(x)$ and for any $\nu > 0$ one considers a family $u_\nu(x,t)$ of Leray–Hopf solutions of the Navier–Stokes equations with the no-slip boundary condition:

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0 \quad \text{and} \quad \nabla \cdot u_\nu = 0 \text{ in } \Omega \times [0,T] \\ \text{with } \; u_\nu = 0 \,\text{on}\, \partial\Omega \times [0,T]. \tag{1.3}$$

For Lipschitz solutions of the Euler equations we have the obvious energy balance relation

$$\int_\Omega \frac{|u(x,t)|^2}{2}\,\mathrm{d}x = \int_\Omega \frac{|u(x,0)|^2}{2}\,\mathrm{d}x, \quad \text{for all } t \in [0,T], \tag{1.4}$$

while for any Leray–Hopf solution of the Navier–Stokes equations we obtain

$$\int_\Omega \frac{|u_\nu(x,t)|^2}{2}\,\mathrm{d}x + \nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \le \int_\Omega \frac{|u_\nu(x,0)|^2}{2}\,\mathrm{d}x, \tag{1.5}$$

for all $t \in [0,T]$.

It is well known that in dimension two the solution u_ν is smooth, unique and (1.5) is actually an equality instead of an inequality. The issue of the regularity of the solutions of (1.3) plays no role in the present contribution which focuses on the zero viscosity limit. It turns out there are no other estimates uniformly valid for all positive ν, and in particular for ν going to zero, other that the one that follows from (1.3). It implies the existence of (may be not unique) limits, in the weak star $L^\infty(0,T;L^2(\Omega))$ topology, of subsequence of solutions u_ν of (1.3). Any such limit is denoted by $\overline{u_\nu}$, and the main question is whether or not we have

$$\overline{u_\nu} = u \;\text{ in } \quad \Omega \times [0,T].$$

As shown in Kato (1972) and Constantin (2005), in the absence of physical boundaries (torus or the whole space) u_ν converges to u.

In the presence of physical boundaries, this is much more subtle. The obvious difficulty comes from the fact that when $\nu \to 0$ only the impermeability boundary condition remains while (here τ denotes the tangential component at the boundary) the relation $(u_\nu)_\tau = 0$ does not persist. Therefore the solution has to become singular near the boundary. It creates a shear flow near the boundary, in solutions of (1.3), which generates vorticity that may propagate inside the domain by the advection term and by the effect of the pressure. This turns out to be the most natural way to generate turbulence (even homogeneous turbulence far from the boundary).

For any Lipschitz vector field w we denote by $S(w)$ its symmetric stress tensor

$$S(w) = \frac{\nabla w + (\nabla w)^{\perp}}{2}.$$

Denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ scalar product. Then since both u_ν and u are divergence-free Lipschitz vector fields and since u is tangent to the boundary of Ω we obtain, by integration by parts, the classical formula

$$(u_\nu \cdot \nabla u_\nu - u \cdot \nabla u, u_\nu - u) = (u_\nu - u, S(u)(u_\nu - u)). \tag{1.6}$$

From (1.2) and (1.3) we also have

$$\partial_t(u_\nu - u) + u_\nu \cdot \nabla u_\nu - u \cdot \nabla u - \nu \Delta u_\nu + \nabla p_\nu - \nabla p = 0. \tag{1.7}$$

Taking the $L^2(\Omega)$ inner product of (1.7) with $(u_\nu - u)$ and observing that on the boundary Ω we have $u_\nu = 0$ and $u \cdot \vec{n} = 0$, thanks to (1.6), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u_\nu - u|^2_{L^2(\Omega)} + \nu \int_\Omega |\nabla u_\nu|^2 \,\mathrm{d}x \\ \le |(u_\nu - u, S(u)(u_\nu - u))| + \nu \int_\Omega (\nabla u_\nu : \nabla u)\,\mathrm{d}x \\ - \nu \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu)_\tau \cdot u \,\mathrm{d}\sigma. \end{aligned} \tag{1.8}$$

The analysis of the term

$$-\nu \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu)_\tau \cdot u \,\mathrm{d}\sigma,$$

which appears in the right-hand side of (1.8) is, in this section and in the next one, the cornerstone of this contribution. We observe that $(\partial_{\vec{n}} u_\nu)_\tau$ is the tangential component of the stress at the boundary. It creates a shear flow near the boundary and generates vorticity. In order to see this more clearly notice that since $(u_\nu)_\tau = 0$ on the boundary of Ω we obtain the following equality

$$-(\partial_{\vec{n}} u_\nu)_\tau \cdot u = (\nabla \wedge u_\nu) \cdot (\vec{n} \wedge u). \tag{1.9}$$

Therefore all the considerations concerning the left hand-side of (1.9) do have their counterpart on the right-hand side, i.e. in terms of the trace of the vorticity of u_ν on $\partial\Omega$.

Moreover, from (1.8) it follows the very easy, but essential result.

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Excerpt
More Information

Proposition 1.2 *Let u be a Lipschitz solution of the Euler equations* (1.2) *and u_ν the solutions of the Navier–Stokes equations* (1.3) *with initial data $u_\nu(x,0) = u(x,0) = u_0(x)$. Then under the hypothesis*

$$\begin{aligned}\limsup_{\nu\to 0} \int_0^T -\nu \int_{\partial\Omega} (\partial_{\vec{n}} \cdot u_\nu)_\tau u \, d\sigma \, dt \\ = \limsup_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} -((\partial_{\vec{n}} u_\nu)_\tau \cdot u_\tau)_- \, d\sigma \, dt \le 0\end{aligned} \tag{1.10}$$

any weak limit $\overline{u_\nu}$ coincides with u in $\Omega \times [0,T]$.

In the proposition and throughout the paper we use $(X)_- = \inf(0, X)$.

Proof From (1.8), using the Cauchy–Schwarz and Young inequalities we deduce that

$$\begin{aligned}|u_\nu - u|^2_{L^2(\Omega)}(t) + \nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 \, dx \, ds \\ \le \nu \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds \\ + 2|S(u)|_{L^\infty(\Omega\times[0,T])} \int_0^t |(u_\nu - u)(s)|^2_{L^2(\Omega)} \, ds \\ + 2 \int_0^t -\nu \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu)_\tau \cdot u \, d\sigma \, ds.\end{aligned} \tag{1.11}$$

Then, under the hypothesis (1.10), we have

$$\begin{aligned}\limsup_{\nu\to 0} |(u_\nu - u)(t)|^2_{L^2(\Omega)} \\ \le |S(u)|_{L^\infty(\Omega\times[0,T])} \int_0^t \limsup_{\nu\to 0} |(u_\nu - u)(s)|^2_{L^2(\Omega)} \, ds,\end{aligned} \tag{1.12}$$

which implies, by Gronwall's inequality, that

$$\limsup_{\nu\to 0} |(u_\nu - u)(t)|^2_{L^2(\Omega)} = 0, \quad \text{for all } t \in [0,T],$$

and consequently, the relation

$$|\overline{u_\nu - u}|^2_{L^2(\Omega)}(t) \le \limsup_{\nu\to 0} |(u_\nu - u)(t)|^2_{L^2(\Omega)} \tag{1.13}$$

implies $\overline{u_\nu} = u$ in $\Omega \times [0,T]$. □

1.2 Kato criterion for convergence to the regular solution.

In a remarkable paper Kato (1984) related the convergence to the smooth solution of the Euler equations to the absence of anomalous energy dissipation in a boundary layer of size ν. At present it turns out that this criterion (this is the object of the Theorem 1.3 below) has several equivalent forms (see Theorem 4.1 in Bardos & Titi (2013) and Constantin et al (2018) for more references). Some of these equivalent forms (in particular the above hypothesis (1.10)) have natural physical interpretations.

Theorem 1.3 *Assume the existence of a Lipschitz solution $u(x,t)$ of the incompressible Euler equations in $\Omega\times]0,T[$. Let $u_\nu(x,t)$ be a Leray–Hopf weak solution of the Navier–Stokes equations (1.3) with no slip boundary condition, that coincides with u at the time $t=0$. Define the region*

$$\mathcal{U}_\nu = \Omega \cap \{d(x,\partial\Omega) < \nu\}.$$

Then the following facts are equivalent:

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} ((\partial_{\vec{n}} u_\nu)_\tau \cdot u_\tau)_- \, \mathrm{d}\sigma \, \mathrm{d}t = 0, \tag{1.14a}$$

$$u_\nu(t) \to u(t) \ \textit{in} \ L^2(\Omega) \ \textit{uniformly in} \ t \in [0,T], \tag{1.14b}$$

$$u_\nu(t) \to u(t) \ \textit{weakly in} \ L^2(\Omega) \ \textit{for each} \ t \in [0,T], \tag{1.14c}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\Omega} |\nabla u_\nu(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{1.14d}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\mathcal{U}_\nu} |\nabla u_\nu(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{1.14e}$$

$$\lim_{\nu\to 0} \frac{1}{\nu} \int_0^T \int_{\mathcal{U}_\nu} |(u_\nu(x,t))_\tau|^2 \, \mathrm{d}x \, \mathrm{d}t = 0, \ \textit{and} \tag{1.14f}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau \cdot w(\sigma,t) \, \mathrm{d}\sigma \, \mathrm{d}t = 0 \tag{1.14g}$$

for all $w(x,t) \in Lip(\partial\Omega \times [0,T])$ tangent to $\partial\Omega$.

Proof The proof is an updated version (cf. Bardos & Titi, 2013) of the basic result of Kato (1984). First observe that (1.14a) is (with $w = u$) a direct consequence of (1.14g).

The fact that (1.14a) implies (1.14b) was observed in the previous section, while (1.14c) clearly follows from (1.14b).

From (1.14c), for any $0 < t < T$, we deduce

$$\begin{aligned} \lim_{\nu\to 0} 2\nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \\ \le \int_\Omega |u(x,0)|^2 \,\mathrm{d}x - \liminf_{\nu\to 0} \int_\Omega |u_\nu(x,t)|^2 \,\mathrm{d}x \\ \le \int_\Omega |u(x,0)|^2 \,\mathrm{d}x - \int_\Omega |u(x,t)|^2 \,\mathrm{d}x \le 0, \end{aligned} \tag{1.15}$$

which gives (1.14d) from which (1.14e) easily follows, as $\mathcal{U}_\nu \subset \Omega$.

Since $u_\nu = 0$ on $\partial\Omega\times]0,T[$ the estimate (1.14f) is deduced from (1.14e) using the Poincaré inequality.

The only non trivial statement is the fact that (1.14f) implies (1.14g) and its proof is inspired by the construction of Kato (1984). We introduce a cut-off function

$$\Theta \in C^\infty(\mathbb{R}),\ \text{with}\ \Theta(0) = 1 \quad \text{and} \quad \Theta(s) = 0 \text{ for } s > 1. \tag{1.16}$$

Then, with $\nu < \eta_0$, use Proposition 1.1 to extend w to a Lipschitz, divergence-free, tangent to the boundary vector field $\hat{w}_\nu$ according to the formula:

$$\begin{aligned} &\hat{w}_\nu(x,t) = 0,\ \text{for} \quad x \notin \mathcal{U}_\nu, \\ &\hat{w}_\nu(x,t) = \nabla \wedge (\vec{n}(\sigma) \wedge w(\sigma,t) d(x,\partial\Omega)\Theta(\frac{d(x,\partial\Omega)}{\nu})), \\ &\qquad\qquad \text{for } x = \sigma(x) - d(x,\partial\Omega)\vec{n}(\sigma(x)) \in \mathcal{U}_\nu. \end{aligned} \tag{1.17}$$

Multiplication of the Navier–Stokes equation satisfied by u_ν and integrating by part gives

$$\begin{aligned} \nu \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau w(\sigma,t)\,\mathrm{d}\sigma \\ = \nu(\nabla u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} - (u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} + (\partial_t u_\nu, \hat{w}_\nu)_{L^2(\Omega)}\,. \end{aligned} \tag{1.18}$$

To show that the right-hand side of (1.18) goes to 0 with ν observe that, the only non trivial terms to consider are those that contain the highest power of ν^{-1}.

We have the following estimates, where C denotes any constant which depends on the geometry and on the Jacobian of the transformation defined on $\mathcal{U}_\nu$ by the relation $x = \sigma(x) - d(x, \partial\Omega)\vec{n}(\sigma(x))$.

$$
\begin{aligned}
\nu \Big| \int_0^T \int_{\mathcal{U}_\nu} |(\nabla u_\nu, \nabla \hat{w}_\nu)| \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&= -\nu \Big| \int_0^T \int_{\mathcal{U}_\nu} u_\nu : \Delta \hat{w}_\nu \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le \nu C \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)||w(\sigma)| \frac{s}{\nu^3} |\Theta'''(\frac{s}{\nu})| \,\mathrm{d}s\,\mathrm{d}\sigma\mathrm{d}t \\
&\quad + o(\nu)
\end{aligned}
\tag{1.19}
$$

and

$$
\begin{aligned}
&\Big| \int_0^T \int_{\mathcal{U}_\nu} (u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} \,\mathrm{d}t \Big| \\
&\le \Big| \int_0^T \int_{\mathcal{U}^\nu} ((u_\nu)_\tau (u_\nu)_n \partial_n (\hat{w}_\tau) \,\mathrm{d}x\,\mathrm{d}t \Big| + o(\nu) \\
&\le C \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)||(u_\nu)_n(\sigma, s)||w(\sigma, t)| \frac{s}{\nu^2} \Theta''(\frac{s}{\nu}) \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\quad + o\,(\nu).
\end{aligned}
\tag{1.20}
$$

Therefore using Cauchy–Schwarz we obtain from (1.19)

$$
\begin{aligned}
\Big| \nu \int_0^T \int_{\mathcal{U}_\nu} (\nabla u_\nu, \nabla \hat{w}_\nu) \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le C \frac{1}{\nu^2} \left(\int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^T \int_{\partial\Omega} \int_0^\nu s^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\le C \left(\frac{1}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \right)^{\frac{1}{2}}
\end{aligned}
\tag{1.21}
$$

and similarly for (1.20) we have

$$
\begin{aligned}
&\left|\int_0^T \int_{\mathcal{U}_\nu} |(u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)| \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leq \left|\int_0^T \int_{\mathcal{U}^\nu} ((u_\nu)_\tau (u_\nu)_n \partial_n(\hat{w}_\tau) \,\mathrm{d}x\,\mathrm{d}t\right| + o(\nu) \\
&\leq C \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma,s)||(u_\nu)_n(\sigma,s)||w(\sigma,t)| \frac{s}{\nu^2}\Theta''(\frac{s}{\nu}) \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad + o(\nu) \\
&\leq \frac{C}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma,s)||(u_\nu)_n(\sigma,s)||w(\sigma,t)| \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t + o(\nu) \\
&\leq C \left(\frac{1}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma,s)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\mathrm{d}t\right)^{\frac12} \\
&\qquad\qquad \times \left(\frac{1}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_n(\sigma,s)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t\right)^{\frac12} + o(\nu).
\end{aligned}
\tag{1.22}
$$

Moreover, since $u_\nu = 0$ on $\partial\Omega$, with the Poincaré inequality, we have

$$
\begin{aligned}
&\int_0^T \int_0^\nu \int_{\partial\Omega} |(u_\nu)_n(\sigma,s,t)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad \leq \nu^2 \int_0^T \int_0^\nu \int_{\partial\Omega} |(u_\nu)_n|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \leq C\|u_0(x)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{1.23}
$$

Therefore the last term of both (1.21) and (1.22) is uniformly bounded by

$$
C\frac{1}{\nu}\int_0^T \int_{\mathcal{U}_\nu} |(u_\nu(x,t))_\tau|^2 \,\mathrm{d}x\,\mathrm{d}t + o(\nu)
$$

and this shows that (1.14f) implies (1.14g), completing the proof. □

1.3 Mathematical and physical interpretation of Theorem 1.3

1.3.1 Recirculation

Since $u_\nu = 0$ on $\partial\Omega$ and u is tangent to the boundary, the fact that

$$
(\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau u_\tau = ((\nabla \wedge u_\nu) \wedge \vec{n}) \cdot u < 0
$$

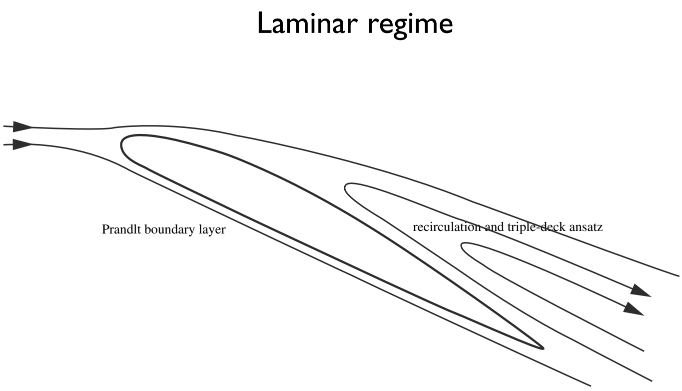

Figure 1.1 Laminar flow with recirculation around an airfoil.

for ν small enough, indicates that somewhere near the boundary the viscous flows u_ν go in the opposite direction to the base flow u that solves the Euler equations, or equivalently that this flow exhibits some backward vorticity. This configuration is known as "recirculation" and does not prevent the fluid from remaining laminar or from having an asymptotic behavior given by the Euler equations, as long as such recirculation is not too big. And this moderate recirculation, shown in Figure 1.1, corresponds to the hypothesis (1.10).

1.3.2 The Prandtl equations and the Stewartson triple-deck ansatz.

As already observed, in the zero-viscosity limit, the boundary condition $(u_\nu)_\tau = 0$ may not persist; hence some type of singularity has to appear near the boundary. However, for linear parabolic problems of the form

$$\partial_t u_\nu - \nu \Delta u_\nu = 0, \qquad u_\nu(x,0) = u_0(x), \qquad u_\nu(x,t)_{|\partial\Omega} = 0 \qquad (1.24)$$

and also, for the linearised Navier–Stokes equations (cf. Ding & Jiang, 2018), the solution converges strongly away from the boundary and near the boundary, in a layer $B_{\sqrt{\nu}} = \{x \in \Omega, d(x,\partial\Omega) < \sqrt{\nu}\}$ of size $\sqrt{\nu}$. It can be described in the laminar regime by a "parabolic scaling", that is,