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978-1-108-45643-2 — Mathematical Structuralism
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# 1 Introduction

This section provides a sketch of structuralism in the philosophy of mathematics, focusing on features shared by all (or most) of the structuralist views in a wide philosophical context. We then provide a list of questions and criteria on which the various structuralist philosophies will be evaluated in subsequent sections.

## Overview

The theme of structuralism is that what matters to a mathematical theory is not the internal nature of its objects – numbers, functions, functionals, points, regions, sets, etc. – but how those objects relate to each other. The orientation grew from relatively recent developments within mathematics, notably toward the end of the nineteenth century and continuing through the present, particularly (but not exclusively) in the program of categorical foundations. Some of the relevant history is recounted in Section 3.

Mathematical structuralism is similar, in some ways, to functionalist views in, for example, philosophy of mind. A functional definition of a mental concept, such as *belief* or *desire*, is, in effect, a structural one, since it, too, focuses almost exclusively on relations that certain items have to each other. The difference is that mathematical structures are more abstract, and free-standing, in the sense that there are no restrictions on the kind of things that can exemplify them (see Shapiro 1997, Chapter 3, §6).

There are a number of mutually incompatible ways to articulate the structuralist theme, invoking various ontological and epistemic theses. Some philosophers postulate a robust ontology of structures, and their places, and then claim that the subject matter of a given branch of mathematics is a particular structure, or a class of structures. An advocate of a view like this would articulate what a structure is, and then say something about the metaphysical nature of structures, and how they and their properties can become known. There are also versions of structuralism amenable to those who deny the existence of distinctively mathematical objects altogether. And there are versions of structuralism in between, postulating an ontology for mathematics, but not a specific realm of structures.

Define a *system* to be a collection of objects together with certain relations on those objects. An extended family is a system of people under certain blood and marital relations – father, aunt, great niece, son-in-law, etc. A work of music is a collection of notes under certain temporal and other musical relations. To get closer to mathematics, define a *natural number system* to be a countably infinite collection of objects with a designated initial object, and a one-to-one successor

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relation that satisfies the axioms of second-order arithmetic (including the second-order induction axiom). Examples of natural number systems are the Arabic numerals in their natural order; a countably infinite sequence of distinct moments of time, say one second apart, in temporal order; the strings on a finite (or countable) alphabet arranged in lexical order; and, perhaps, the natural numbers themselves. Define a *Euclidean system* to be three collections of objects, one to be called "points," a second to be called "lines," and a third to be called "planes," along with certain relations between them, such that the Euclidean axioms are true of those objects and relations, so construed.

A *structure* is the abstract form of a system, which ignores or abstracts away from any features of the objects that do not bear on the relations. So the natural number structure is the form common to all of the natural number systems. The Euclidean structure is the form common to all Euclidean systems, etc.

A structure is thus a "one over many," a sort of universal. The main difference between a structure and a more traditional universal, such as a property, is that a given property applies to, or holds of, individual objects, while a given structure applies to, or holds of, entire *systems*.

Any of the usual array of philosophical views on universals can be adapted to structures, thus giving rise to some of the varieties of structuralism. One can be a Platonic *ante rem* realist about structures, holding that each structure exists and has its properties independent of any systems that have that structure – or at least independent of those systems that are not themselves structures. We call this view *sui generis structuralism* (SGS). On this view, structures exist objectively, and are ontologically prior to any systems that have them (or at least they are ontologically independent of such systems). Or one can be an Aristotelian *in re* realist, holding that structures exist, but insisting that they are ontologically posterior to the systems that instantiate them. One variety of this *in re* view is what we call *set-theoretic structuralism* (STS). On that view, structures are isomorphism types (or representatives thereof) within the set-theoretic hierarchy. The distinction between these two kinds of realism raises metaphysical issues of grounding and ontological priority.

Another option is to deny that structures exist at all. Talk of a given structure is just convenient shorthand for talk of all systems that are isomorphic to each other, in the relevant ways. Views like this are sometimes called *eliminative* structuralism, since they eschew the existence of structures altogether.

Advocates of the different ontological positions concerning structures take different approaches to other central philosophical concerns, such as epistemology, semantics, and methodology. Each such view has it relatively easy with some issues and finds deep, perhaps intractable problems with others. The *ante rem* SGS view, for example, has a straightforward

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account of reference and of the semantics of the languages of mathematics: the variables of a branch of mathematics, such as arithmetic, real analysis, or complex analysis, range over the places in an *ante rem* structure. Singular terms denote individual places, so the language is understood at face value.

In other words, an advocate of SGS has it that the straightforward grammatical structure of a mathematical language reflects the underlying logical form of the propositions. For example, in the simple arithmetic equation, $3 \times 8 = 24$, the numerals '3', '8', and '24' at least seem to be singular terms – proper names. In the SGS view, they *are* singular terms. The role of a singular term is to denote an individual object and, in the SGS view, each of these numerals denotes a place in the natural number structure. And, of course, the equation expresses a truth about that structure. In this respect, then, SGS is a variation on traditional Platonism. For this perspective to make sense, however, one has to think of a place in a structure as a *bona fide* object, the sort of thing that can be denoted by a singular term, and the sort of thing that can be in the range of first-order variables.

An advocate of the SGS approach agrees with the eliminativist (and the *in re* realist) that mathematical statements in, say, arithmetic, *imply* generalizations concerning systems that exemplify the structure. We say, for example, that in any natural number system, the object in the three-place multiplied (using the relevant relation in the system) by the object in the eight-place is the object in the twenty-four-place. Of course, the generalizations themselves do not entail that there are any natural number systems – nor any *ante rem* structures for that matter.

The eliminativist holds that mathematical statements *just are* (or are best interpreted as) generalizations like these, and she accuses the SG structuralist of making too much of their surface grammar, trying to draw deep metaphysical conclusions from that. For example, the simple theorem of arithmetic, "for every natural number $n$ there is a prime $p > n$" is rendered:

*In any natural number system S, for every object x in S, there is another object y in S such that y comes after x in S and y has no divisors in S other than itself and the unit object of S.*

In general, any sentence $\Phi$ in the language of arithmetic gets regimented as something like the following:

$$\text{In any natural number system } S, \ \Phi[S], \qquad (\Phi')$$

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where $\Phi[S]$ is obtained from $\Phi$ by restricting the quantifiers to the objects in $S$, and interpreting the non-logical terminology in terms of the relations of $S$.

In a similar manner, the eliminative structuralist paraphrases or regiments – and deflates – what seem to be substantial metaphysical statements, the very statements made by her SGS opponent. For example, "the number 2 exists" becomes "in every natural number system $S$, there is an object in the 2-place of $S$"; or "real numbers exist" becomes "every real number system has objects in its places." These statements are trivially true – analytic, if you will – not the sort of statements that generate heated metaphysical arguments.

However, the sailing is not completely smooth for the eliminativist. Suppose, for example, that the entire physical universe consists of no more than $10^{100,000}$ objects. Then there are no natural number systems (since each such system must have infinitely many objects). So for *any* sentence $\Phi$ in the language of arithmetic, the regimented sentence $\Phi'$ is vacuously true. So the eliminativist would be committed to the truth of (the regimented version of) $1 + 1 = 0$.

In other words, a straightforward, successful eliminative account of arithmetic requires a countably infinite background ontology. And it gets worse for other branches of mathematics. An eliminative account of real analysis demands an ontology whose size is that of the continuum; for functional analysis, we would need the power set of that many objects. And on it goes. The sizes of some of the structures studied in mathematics are staggering.

Even if the physical universe does exceed $10^{100,000}$ objects, and, indeed, even if it is infinite, there is surely *some* limit to how many physical objects there are. So, for the eliminative structuralist, branches of mathematics that, read at face value, require more objects than the number of physical objects end up being vacuously trivial. This would be bad news for such theorists, as the goal is to make sense of mathematics as practiced. In any case, no philosophy of mathematics should be hostage to empirical and contingent facts, including the number of objects in the physical universe.

There are two eliminativist reactions to this threat of vacuity. First, the philosopher might argue, or assume, that there are enough *abstract* objects for every mathematical structure to be exemplified. In other words, we postulate that, for each field of mathematics, there are enough abstract objects to keep the regimented statements from becoming vacuous.

Some mathematicians, and some philosophers, think of the set-theoretic hierarchy as the ontology for all of mathematics. Mathematical objects – all mathematical objects – are sets in the iterative hierarchy. Less controversially, it is often thought that the iterative hierarchy is rich enough to recapitulate every mathematical theory.

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An eliminative structuralist might maintain that the theory of the background ontology for mathematics – set theory or some other – is not, after all, the theory of a particular structure. The foundation is a mathematical theory with an intended ontology in the usual, non-structuralist sense. In the case of set theory, the intended ontology is the sets. Set theory is not (merely) about all set-theoretic systems – all systems that satisfy the axioms. So the foundational theory is an exception to the theme of structuralism. But, the argument continues, every *other* branch of mathematics is to be understood in eliminative structuralist terms. This is the route of what we call STS.

Of course, this ontological version of eliminative structuralism is anathema to a nominalist, who rejects the existence of *abstracta* altogether. For the nominalist, sets and *ante rem* structures are pretty much on a par – neither is wanted. The other prominent eliminative reaction to the threat of vacuity is to invoke modality. In effect, one avoids (or attempts to avoid) a commitment to a vast ontology by inserting modal operators into the regimented generalizations. To reiterate the above example, the modal eliminativist renders "for every natural number $n$ there is a prime $p > n$" as something like:

*In any possible natural number system S, for every object x in S, there is another object y in S such that y comes after x in S and y has no divisors in S other than itself and the unit object of S.*

In general, let $\Phi$ be any sentence in the language of arithmetic; $\Phi$ gets regimented as:

In any possible natural number system $S, \Phi[S]$,

or, perhaps,

Necessarily, in any natural number system $S$, $\Phi[S]$,

where, again, $\Phi[S]$ is obtained from $\Phi$ by restricting the quantifiers to the objects in $S$, and interpreting the non-logical terminology in terms of the relations of $S$.

The modal structuralist also asserts that the various systems of mathematics are possible. It is possible for there to be a natural number system, a real number system, a Euclidean system, etc.

The difference with the ontological, eliminative program, of course, is that here the variables ranging over systems are inside the scope of a modal operator. So the modal eliminativist does not require an extensive, rich background ontology. Rather, she needs a large ontology to be possible.

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The central problem with this brand of eliminativist structuralism concerns the nature of the invoked modality. Of course, it will not do much good to render the modality in terms of possible worlds. If one does that and takes possible worlds, and *possibilia*, to exist, then modal eliminative structuralism would collapse into the above ontological version of eliminative structuralism. Not much would be gained by adding the modal operators. The modalist typically takes the modality to be *primitive* – not defined in terms of anything more fundamental. But, of course, this move does not relieve the modalist of having to say something about the nature of the indicated modality, and something about how we know propositions about what is possible. We develop two versions of modal structuralism in subsequent sections, with references to the literature.

To briefly sum up and conclude, the parties to the debate over how best to articulate the structuralist insights agree that each of the major versions has its strengths and its peculiar difficulties. Negotiating such trade-offs is a stock feature of philosophy. The literature has produced an increased understanding of mathematics, of the relevant philosophical issues, and how the issues bear on each other.

## The State of the Economy

We plan to evaluate each version of structuralism by considering how well it fares on each of the following eight criteria.

(1) What primitives are assumed and what is the background logic? Is it just first-order logic, or is second- or higher-order logic employed? If the latter, what is the status of relations and functions? What advantages and limitations are implied by these various choices?

(2) The term "axioms" is ambiguous, as between "defining conditions on a type of structure of interest," on the one hand, and "basic assumptions" or "assertoric content," bearing a truth-value, on the other. It is characteristic of a structuralist view of mathematics to emphasize axioms in the former sense, as defining conditions on structures of interest; and this was the sense in which, for instance, Dedekind (1888) introduced the so-called "Peano postulates" on the natural number system in his classic essay, and it was axioms in this sense that Hilbert invoked in his well-known correspondence with Frege, who emphasized axioms in the assertoric sense (see the next section).

One should recognize that Frege had a point, viz. that a foundational framework requires some assertory axioms, capable of being true or false, governing especially the existence and nature of structures. When

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it comes to the different forms of structuralism, what are these "assertory" axioms?

(3) As an especially important case of (2), what assumptions are asserted as to the mathematical existence of structures? Is their indefinite extendability recognized or is there commitment to an absolutely maximal universe?

(4) Are structures recognized as a special type of object, or is there a thoroughgoing *elimination* of structures as objects? If not, what sort of objects are "structures," and, in particular, what is a *mathematical structure*?

(5) How is our epistemic access to structures understood, and what account of reference to them can be given?

(6) As an extension of (5), does the view allow for a face-value interpretation of mathematical statements? For example, do what appear to be singular terms in the languages of mathematics get rendered as singular terms? Do quantifiers get rendered as straightforward quantifiers? Or is there some regimentation or paraphrase involved?

(7) How are the paradoxes associated with set-theory and other foundational frameworks (such as category theory) to be resolved?

(8) Finally, how is Benacerraf's challenge based on competing identifications of numbers, etc., to be met?

## 2 Historical Background

Howard Stein (1988, p. 238) claims that during the nineteenth century, mathematics underwent "a transformation so profound that it is not too much to call it a second birth of the subject" – the first birth being in ancient Greece. The same period also saw important developments in philosophy, with mathematics as a central case study.

According to Alberto Coffa (1991, p. 7), for "better or worse, almost every philosophical development since 1800 has been a response to Kant." A main item on the agenda was to account for the *prima facie* necessity of mathematical propositions, the *a priori* nature of mathematical knowledge, and the applicability of mathematics to the physical world, all without invoking Kantian intuition. Can we understand mathematics independent of the forms of spatial and temporal intuition?

Coffa argues that the most successful approach to this problem was that of what he calls the " semantic tradition," running through the work of Bernard Bolzano, Gottlob Frege, the early Wittgenstein, and David Hilbert, culminating with the Vienna Circle, notably Moritz Schlick and Rudolf Carnap. The plan was

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to understand necessity and a priority in *formal* terms. In one way or another, this philosophical tradition was linked to the developments in mathematics. One legacy left by the developments in both mathematics and philosophy is mathematical logic, and model-theoretic semantics in particular. The emergence of model theory and the emergence of structuralism are, in a sense, the same.

In this section, we recount some themes in the development of Euclidean, projective, and non-Euclidean geometry, as well as some themes in arithmetic. Concerning geometry, there was a gradual transformation from the study of absolute or perceived space – matter and extension – to the study of structures. Our narrative includes sketches of early-twentieth-century theorists who either developed structuralist insights, or opposed these moves, or both. The list includes Dedekind, Frege, and Hilbert, among others.[1]

## Geometry, Space, Structure

The historical transition away from geometry as the study of physical or perceived space is complex. One early theme is the advent and success of analytic geometry, with projective geometry as a response. Another is the attempt to accommodate ideal and imaginary elements, such as points at infinity. A third thread is the assimilation of non-Euclidean geometry into mainstream mathematics (and into physics). These themes contributed to a growing interest in rigor and the eventual detailed understanding of rigorous deduction as independent of content – ultimately to a structuralist understanding of mathematics. Here, we can provide no more than a mere sketch of a scratch of this rich and wonderful history.

The traditional view of geometry is that its subject matter is matter and extension. The truths of geometry seem to be necessary, and yet geometry has something to do with the relations between physical bodies. Kant's account of geometry as synthetic *a priori*, relating to the forms of perceptual intuition, was a heroic attempt to accommodate the necessity, the *a priori* nature, *and* the empirical applicability of geometry.

The traditional view of arithmetic is that its subject matter is quantity. Arithmetic was the study of the discrete, while geometry was the study of the continuous. The fields were united under the rubric of mathematics, but one might wonder what they have in common other than this undescribed genus. The development of analytic geometry went some way toward loosening the

[1] Much of this section draws from Shapiro 1997, Chapter 5), used with kind permission from Oxford Universith Press, as well as Nagel (1939), Freudenthal (1962), Coffa (1986; 1991, Chapters 3 and 7), and Wilson (1992). Readers interested in these episodes of mathematical history are urged to consult those excellent works.

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distinction between them. Mathematicians discovered that the study of quantity can shed light on matter and extension (and vice versa).

One result of the development of analytic geometry was that synthetic geometry, with its reliance on diagrams, fell into neglect. Joseph-Louis Lagrange even boasted that his celebrated treatise on mechanics did not contain a single diagram (but one might wonder whether his readers appreciated this feature). The dominance of analytic geometry left a void that affected important engineering projects. For example, problems with plane representations of three-dimensional figures were not tackled by mathematicians. The engineering gap was filled by the emergence of projective geometry (see Nagel 1939, §§7–8). Roughly, projective geometry concerns spatial relations that do not depend on fixed distances and magnitudes, nor on congruence. In particular, projective geometry dispenses with quantitative elements, like a metric.

Although all geometers continued to identify their subject matter as intuitable, visualizable figures in space, the introduction of so-called ideal elements, such as imaginary points, into projective geometry constituted an important move away from visualization. Parallel lines were thought to intersect, at a "point at infinity," although, of course, no one can visualize that, in any literal sense. Girard Desargues proposed that the conic sections – circle, ellipse, parabola, and hyperbola – form a single family of curves, since they are all projections of a common figure from a single "improper point" – located at infinity. Circles that do not intersect in the real plane were thought to have a pair of imaginary points of intersection. As Ernest Nagel (1939) put it, the "consequences for geometrical techniques were important, startling, and to some geometers rather disquieting" (p. 150). Clearly, mathematicians could not rely on the forms of perceptual intuition when dealing with the new imaginary elements. The elements are not in perceivable space; we do not *see* anything like them.

The introduction and use of imaginary elements in analytic and projective geometry were an outgrowth of the development of negative, transcendental, and imaginary numbers in arithmetic and analysis. With the clarity of hindsight, there are essentially three ways that "new" entities have been incorporated into mathematical theories (see Nagel 1979). One is to *postulate* the existence of mathematical entities that obey certain laws, most of which are valid for other, accepted entities. For example, one can think of complex numbers as like real numbers but closed under the taking of roots, and one can think of ideal points as like real points but not located in the same places. Of course, postulation begs the question against anyone who has doubts about the entities. Recall Bertrand Russell's ([1919] 1993, p. 71) quip about how postulation has the advantages of theft over honest toil.

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In reply, one might point to the *usefulness* of the new entities, especially for obtaining results about established mathematical objects. But this benefit can be obtained with *any* system that obeys the stipulated laws. Thus, the second method is *implicit definition*. The mathematician gives a description of the system of entities, usually by specifying its laws, and then asserts that the description applies to any collection that obeys the stipulated laws. At this point, the skeptic might wonder whether there are any systems of entities that obey the stipulated laws.

The third method is construction, where the mathematician defines the new entities as combinations of already established objects. Presumably, this is the safest method since it settles the question of whether the entities exist (assuming the already-established objects do). William Rowan Hamilton's definition of complex numbers as pairs of real numbers fits this mold as does the logicist definition of natural numbers as collections of properties. A fruitful outlook would be to take implicit definition and construction in tandem. A construction of a system of objects establishes that there are systems of objects so defined, and so the implicit definition is not empty. Moreover, the construction also shows how the new entities can be related to the more established ones and may suggest new directions for research.

Nagel (1979) notes that all three methods were employed in the development of ideal points and points at infinity in geometry. Jean-Victor Poncelet came close to the method of postulation. In trying to explain the usefulness of complex numbers in obtaining results about the real numbers, he claimed that mathematical reasoning can be thought of as a mechanical operation with abstract signs. The results of such reasoning do not depend on any possible referents of the signs, so long as the rules are followed. Having thus " justified" new sorts of numbers in analysis, Poncelet went on to argue that geometry is equally entitled to employ abstract signs – with the same freedom from interpretation. He held that traditional, synthetic geometry is crippled by the insistence that everything be cast in terms of drawn or visualizable diagrams.

Poncelet's contemporaries were aware of the shortcomings of such bare postulation. Nagel cites authors like Joseph Diaz Gergonne and Hermann Grassmann, who more or less prefigured the method of implicit definition. Their work furthered the concern with rigor and the abandonment of the traditional view of geometry as concerned with extension. We move closer to a structuralist perspective on geometry. Grassmann's *Ausdehnungslehre* of 1844 developed geometry as "the general science of pure forms," considered in abstraction of any interpretation the language may have. He characterized the terms of geometry only by stipulated relations they have to each other:

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