# 1 Introduction

## 1.1 Aims and Sources

Few have characterized Wittgenstein's philosophy of mathematics over the entirety of his intellectual life. Some favor the Early and Middle Wittgenstein *over* the Later. Below I argue that Later Wittgenstein, a dialectical revisor of Early and Middle Wittgenstein, offers us a defensible contemporary philosophy of mathematics.

In 1944 Wittgenstein wrote that his "chief contribution has been in the philosophy of mathematics."[1] He aimed at a book whose first part would clarify the nature of meaning and whose second part would apply that clarification to the foundations of logic and mathematics. RFM is an edited selection from unfinished drafts Wittgenstein hoped would form PI part 2. But he withheld its publication. These writings utilize Wittgenstein's Later interlocutory style, but lack polished orchestration; interpreters have frequently distorted his ideas by using the method of drive-by quotation. Selected remarks lack their original context. The tentativeness of Wittgenstein's thoughts is obscured. Since 2000, BEE has been used to supplement RFM with manuscripts. We rely on commentaries using BEE.[2]

## 1.2 Aspect Realism

Wittgenstein sees mathematicians articulating conceptual constructions that provide standpoints for modeling empirical and mathematical facts, situations, structures, events, procedures, and characterizations. While its techniques are intertwined with logical features of language, mathematics is not reducible to a single "foundation" such as second-order logic or set theory. Later Wittgenstein suggests looking at mathematics as a "MULTICOLORED mix of techniques" (RFM III §§46,48) to nuance the kind of anti-foundationalism he always urged.

In Wittgenstein's Later Philosophy, aspects are *discovered* or revealed, whereas mathematical techniques are *invented*.[3] Mathematics is *both* discovery and invention. I read Wittgenstein as an "aspect realist" about mathematics.

This is a "realistic" form of realism in the sense of Diamond (1991) and Putnam (1999): realism without grounded metaphysics and no particular epistemology or theory of mind. It transposes what is often called "realism." I shall not worry whether the informal language of "seeing aspects" is metaphorical

[1] Monk, 1990, 466.
[2] Mühlhölzer, 2010 (RFM III); WH (RFM II, V).
[3] BT, §134; RFM II §38, RFM III §§46ff; MS 122, pp. 15, 88-88, 90; PI §§119,124-129,133, 222, 262, 387, and 536; xi, p. 196; PPF xi, §130. Floyd, 2018a; Harrington, Shaw, and Beaney, 2018; and Baz, 2020 concern aspects generally.

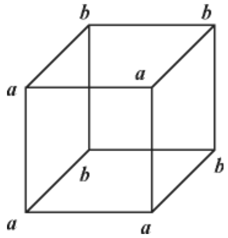


**Figure 1** Necker Cube
**Source:** TLP 5.5423.

or wholly misjudged. The phenomenon is clear enough in the ambiguous depth cues of Jastrow's Duck-Rabbit and the Necker Cube (Figure 1).

Aspects are modal, attaching to possibilities and necessities: fields of significance, opportunities for projecting and instantiating our concepts. We see *through* the picture to our own seeing of it *as* realizing one way among others. What we see is seen, but also we see. We rearticulate what we see, sometimes seeing it thereby anew. There is an active and a passive aspect to this. Aspects show themselves (the middle voice).[4] What we are seeing is not simply an actual drawing on a page. We can also "see" *in* these drawings *possibilities* of projecting our concepts. Here we take modality as primitive, though up for investigation.

Wittgenstein always rejected the "Platonist" idea that mathematics reveals actual entities, abstract facts, or realms that explain our mathematical practices but are causally inert. Aspectual realism is an offered substitute. He has no story about whether a possibility is *itself* a possible state of affairs; he is burying ontology in that sense, as many current philosophers of mathematics do.[5]

Aspects shift the "naturalism" urged in psychologistic, conventionalist, and Humean readings of Later Wittgenstein: these underplay his Later views on truth, confusing them with assertability conditions.[6] Since Benacerraf (1965), many philosophers of mathematics have aimed to balance three main ideas: (a) an epistemological position privileging immediate perception of material objects, conceived as a causally cemented, receptive lining up of experience with words or concepts; (b) phenomena such as incompleteness, undecideability, and nonstandard models; (c) modality as something to be explained in metaphysical terms.

4 Narboux, 2014.
5 Auxier, Anderson, and Hahn, 2015, 11; Putnam, 2004.
6 Fogelin, 1987, 2009; Maddy, 2014.

Wittgenstein rejects these ideas, while respecting their deep appeal. He broached a phenomenological account of mathematics in his early Middle Period as a way to resist them (§3.2). This failed, leading him to reject the idea that experience is intrinsically structured in a life-world. His Later Philosophy seeks to dislodge the argument between the naturalistic, causal view and the phenomenological one: both sides underrate the power of ordinary, colloquial ways of speaking in and about mathematics.

Though logicians have not failed to attend to the role of ordinary language (§3.1), the signature element in every period of Wittgenstein's philosophy is his reflection on our "silent adjustments" of its complexities (TLP 4.002). The Later philosophy finally factors these adjustments into realism itself.

Psychological contingencies are relevant to the question, "What is the nature of logic (and mathematics)?" Psychology explores aspect-perception as a phenomenon. But psychology cannot substitute for critical discussion of our ordinary discussion. When Wittgenstein remarks that a proof must be "surveyable," he is not discussing the width of our cognitive abilities but our need for *mathematics* (§4.4). His concerns about extensionalism (§3.5, §4.5) are in place even though, so far as I know, cognitive psychologists have no idea how to account for our coming to grasp the concept of *set*.

Realistically speaking, in mathematics there is moulting and molding of concepts: truth is not simply a matter of grasping an extension, asserting or stipulating a principle. Aspect "perceptions" show us "the limits of empiricism" in *Wittgenstein*'s sense.[7] This contrasts with typical mathematicians' images of these limits. Russell (1936) argued that the limits lie solely in the "medical" fact of our finitude and the need for abstraction to universals: we cannot drink an infinite number of glasses of water. But this is a contingent fact. Hrbacek and Jech (1999, 86), a textbook in set theory, states, for example, that

> it is possible to write down decimal expansions $0.a_1a_2a_3$ ... where "$\langle a_i \rangle_{i=1\ldots\infty}$" is an arbitrary sequence of integers between 0 and 9.

Most readers pass over such remarks silently. Later Wittgenstein urges, not that such "supertasks" are incoherent, but that the limits of empiricism lie elsewhere: first, in our need to *communicate proofs* (RFM III §71), therefore, second, in the activity of *concept-formation* (RFM IV §29), and third, in our embeddings of words and symbols (including the above-quoted ones) in *forms of life* (RFM VII §§17, 21).

7 RFM III §71, IV §29, VII §§17, 21.

Maddy (2011, 2007) advocates“thin” realism about sets and a Second rather than a First philosophy. “Aspect realism” is a form of this. But Maddy focuses first and foremost on set theory. Moreover, she lacks confidence in our ability to articulate a sense of depth in mathematics. Aspect realism is intended to round out “thin realism” to convey more of a 3D sense of mathematical depth and insight.

“Naturalistic” philosophers of mathematics have tended to privilege physics as the arbiter of ontology, and psychology as the basis for “naturalized epistemology.”[8] In Wittgenstein a more “liberal” form of naturalism is in view, the kind advocated by the post-1990 Putnam.[9] Like Wittgenstein, Putnam held that mathematics explores conceptual possibilities (he called this “modal structuralism”). But he insisted on the normative elements in play here. “Forms of life” express the human animal’s ways of structuring possible lives with language, and this has an evolutionary, biological tint, but also a normative and ethological one.[10]

The Later Wittgenstein emphasizes the importance of projectability and plasticity: the work of fitting concepts *to* reality, including the reality of mathematical (and other) experiences. This form of realism not only glosses the wide and multifarious kinds of applicability of mathematics – to empirical situations, to mathematics itself, to experiences, and to concepts. It also allows mathematics its autonomy, as modal structuralism does. Wittgenstein treats mathematics, however, more thinly: as a kind of *scaffolding* for descriptions, that is, a modular, transportable collection of possible conceptual constructions that may be configured and reconfigured in an unlimited variety of ways. It does not support the edifice of knowledge so much as help human beings erect it.

What we learn in mathematics comes to feel so natural, in certain cases, that it comes to shape our immediate experience, embedding its modalities deeply in our habits and perceptions. There is an echo of Kant here. Wittgenstein does not, however, forward a view of intuition as a fixed form of immediate, singular, nonconceptual representation. The idea instead is that there are mathematical *possibilities* and *necessities*, “forms” of particular, immediate experience, construction, and conceptualization typical of human beings and communicable among them.

Wittgenstein is *grammaticalizing* the “intuitive,” that is, subjecting it to discussion, using the ordinary language of mathematics to do so, taking this language as (an evolving) given. He does not replace mathematical experience

8 Maddy, 1997, 2005.
9 Putnam, 1990, 1992, 2012, 2016.
10 Cavell, 1988.

*with* language, but rather uses language to open up our receptiveness *to* it. "Language itself provides the necessary intuition" (TLP 6.233). This allows him to capture "the reflective element" that Bernays (1959) missed in RFM. In the Later Philosophy, it is as if the whole idea of a metalanguage is absorbed into the ever-shuffling process of formalization, reformulation, renewed experience, informal characterization, re-parametrization, and reinterpretation. We should respect the hustle.

I detect an affinity with a Gödelian norm that plays a crucial role in mathematical practices, lately developed by Kennedy (2020). She calls it *formalism-freeness*. In Wittgenstein this is a matter of characterizing what formalisms are and mean, ideally with a minimal degree of formalization. Both Gödel and Wittgenstein associate aspect-richness with incompleteness. Whereas Gödel hypothesizes that there may be an actual infinity of complexity given in experience, Wittgenstein regards the unbounded degree of complexity as potential and grammaticalizable.[11] He stresses the *non*-extensional rather than the extensional perspective. While Gödel felt the perspectives could be conceptually merged, Wittgenstein resisted this.

I shall use the term "non-extensional" instead of the more usual term "intensional," which is typically associated with constructivism as a working branch of mathematics. I do not regard Wittgenstein as a constructivist about mathematics, and he did not believe in intensions as entities.

Aspect-talk allowed Wittgenstein to soberly discuss the *experience* of novelty, the reorientation of our way of seeing that comes when we encounter mathematical features, forms, and structures anew.[12] There is a two-ness of grammatical complexity in the entanglement of modality with the ways we use the verb to "see" and "seeing-as" phrases to describe it: through characterizations we can come to "see" or "reveal" or "discover" a possible way of conceiving of something we hadn't seen before. Mathematics is filled with such "seeing." This is not literal perception, but more like seeing a new dimension or possibility for thought.

Wittgenstein takes our as- and aspect-phrasing to serve us essentially in our ability to draw meaningful distinctions between characterizations and properties, discoveries and inventions, appearances and realities, possibilities and necessities. Aspects open up space for the *kind* of discoveries it is possible to make in philosophy, logic, and mathematics – and also the kinds of things that people may *miss*. That there is such a thing as insight into things and people, into possibilities that are not necessities, into truths and falsehoods, and

[11] Floyd and Kanamori, 2016, MS 163, 40v.
[12] RFM I App II (Mühlhölzer, 2002) criticize unsober ideas of discovery.

that such things are not merely fleeting appearances or conventions – these are among Wittgenstein's deepest philosophical themes.

*Aspect* is therefore a logical notion. It reworks Frege and Russell. For them, truth is absolute; there is no *way* of being true.[13] For Wittgenstein there are different ways of articulating truth *as* an absolute notion.

The term "aspect" does not occur in Frege's German, though English translators have often used it to render his remarks on the decomposition of judgments. We hear Frege speak (1879, §9) of our "imagining that an expression can be altered" at a place in a sentence; in shifting "$2^4 = 16$" to "$x^4 = 16$" the term is "regarded as replaceable" (Bauer-Mengelberg) and we may "imagine the 2 in the content" so that the content is "split into a constant and a variable part" (1880/1881, 17–19), though Frege actually is careful to say that it divides, or decomposes itself. In

$$(1 + 1) + (1 + 1) + (1 + 1) = 6$$

"the different expressions correspond to different conceptions [aspects] and sides, but nevertheless always to the same thing" (1891, 4f.); a name "illuminates its reference only one-sidedly" (1892, 27).

These imaginings, regardings, and 3D metaphors flirt with procedural and with psychologistic language, even though Frege is marking something static: the generality of a concept. Wittgenstein's aspect-talk picks up on this. The slide is between a metaphor of proceeding, "thinking" a position *as* variable, and a metaphor of something exposed (a concept or function). Frege himself warned against this metaphorical danger (1897, 157n), (1918/19, 66).

In TLP Wittgenstein emphasizes the multiple "standpoints" (aspects as prospects, or views) from which the sameness or difference of meaning of a sentence or arithmetic expression might be assessed (6.2323). Mathematics consists of equations and uses the "method of substitution" (6.234ff.). Logical patterns – shared "internal" features of sentences – evince "faces" or "looks" (4.1221). But Wittgenstein replaces Frege's idea of "varying a position" in a sentence by the idea of possible step-by-step "operations": formal procedures that evince "internal" necessity. He operationalizes generality and meaning.

Like Frege, Wittgenstein resists the idea that the process of decomposing a thought into its components requires "intuition." But he rejects Frege's general sense/reference distinction. For Wittgenstein (Early, Middle and Later), names have *Bedeutung* (reference), but not sense; only sentences, in which names structurally figure, have sense (*Sinn*), evince aspects, help project the realization of specific possible situations.

[13] Shieh, 2019.

Wittgenstein's aspect-phrasings invite the charge that he confused talk about essence with essence, sign with object, truth with appearance. Russell's use of aspects in his (1914) construction of the world had urged something close to this. But Wittgenstein was pursuing the reverse idea, committed to fulfilling the analytical role played by Russell's notion of "acquaintance": immediate, direct knowledge of a mind-independent, particular thought-form. In responding to Russell, Wittgenstein returns the concept of "acquaintance" to its everyday home, the sense in which one may be "acquainted" with a person, a proof, a face, a habitat: one looks, speaks, walks around, thinks, from different angles. This de-emphasizes the causal story by recasting the role of "perception."

For Wittgenstein, mathematics may be said to be "about" numbers, aspects of concepts, and so on, but only in an ordinary language sense familiar from Austin.[14] This emphatically does *not* mean that "all we are talking about when we do mathematics is language" or that Wittgenstein confused linguistic and metaphysical matters. Ordinary language is about whatever we talk about: so it is not *just* the words, it is how we look at things, describe things, *fit* our concepts *to* one another and reality. What is "ordinary" is not simply given; rather, it is what is familiar, taken for granted. It must be thought *through* – occasionally denied or shifted – for its potentialities (aspects) to be seen.

A number, as Gowers nicely puts it, "*is* what it *does*" (2002, 18). But it does not do what it does, have the force and reality and necessities and possibilities it does, without being fit for (indefinite) elaboration in specific ways – like any action, human or animal. An action is no action at all unless it falls under a series of intelligible characterizations. An action, a number, a proof have different *aspects*, that is, necessities, faces, characters.[15]

These phenomena are real. We can regard the real number field *as* a field; a proof *as* a model of an experiment (RFM I); we may see (or miss) a grasp of the concept "counting" *in* a child's behavior. This projection and reassembling of concepts is, in fact, ubiquitous.

## 2 Early Philosophy (1912–1928): Absolute Simplicity

### 2.1 "Final" Analysis

TLP urges an ideal of *absolute* simplicity: the design of a logical notation in which the totality of sentences ("what can be said") is formally constructed, step-by-step, from a base of simplest "elementary" sentences, atomic configurations of "simple" names that "picture" facts. Possibilities of combinations

[14] Mühlhölzer, 2014.
[15] Diamond, 1991; Floyd, 2010, 2012a; Putnam, 2012.

of names in elementary sentences would mirror the possible configurations of facts. Wittgenstein uses schematic language – “$f(x)$”,“$\phi(x, y)$”, “$p$”, and so on (4.24) – but this draws out necessary “formal” features of any language, not specific values in a particular language from the point of view of a metalanguage. There is just *one* language that I understand, just *one* logical space.

Russell broached the possibility of a hierarchy of languages in his Introduction to the TLP. But Wittgenstein never accepted that reasoning in a metalanguage about a syntactically specified formal language clarified fundamental philosophical issues. If devoid of attachment to a working, meaningful language, a formal language is logically accidental. Wittgenstein distinguishes between *signs* (sign types, e.g., “A is the same sign as A” [3.203]) and *symbols* that are essential (logical) features of the expression of thought in any language.

The main task in rewriting ordinary sentences in a Tractarian notation is to separate what is arbitrary from what is not arbitrary in thought. Symbols receive a “formal” expression in the final analysis: it is unthinkable that an object or thought lacks an “internal” feature that its symbols so display (4.1221).

The TLP excludes a “theory” of types: the possibilities of saying what is the case inscribe types directly into the grammatical workings of the symbols. What it is impossible to think is nonsense, and Wittgenstein has no theory of what that is. Russell’s “paradox” is obviated by seeing that a concept such as $F(fx)$ cannot take itself as argument, because if this is done, the working symbolic “prototypes” automatically disambiguate: “$F(F(fx))$” should be rendered $\psi(\phi(fx)$ (3.333).

Wittgenstein’s conception of second-order generalization is unclear. Do types automatically circumscribe the range of sentences demarcated by $\psi(\phi(fx)$? Did he take such a form to be constructible, step-by-step, from the forms of elementary sentences? On one reading we imagine free use of second-order generalization, as in Frege, with types carrying the load.[16] Alternatively only predicatively defined, recursive second-order concepts are allowed in the generation of second-order variables.[17]

An intermediate position envisions a type-free language with a $\lambda$-operator and a cumulative, predicative presentation setting out the language in stages: second-order variables shift their range depending upon the (finite) stage of the construction of the language.[18] This expresses a portion of *Principia*’s ramified type theory: eventually every second-order “form” appears, its instances restricted from below.

16 Potter, 2000; but compare Weiss, 2017, 7.
17 Ricketts, 2014; Sundholm, 1992.
18 Fisher and McCarty, 2016.

Possibilities of logic are not states of affairs that we picture, but come through – are "shown" in – our saying what is the case and our thinking, truly or falsely. We have the capacity to regard our sayings as affirming and/or denying the realization of this or that set of possibilities among others, leaving certain possibilities aside as logically independent. Logic explores the formal interrelations among (structured) sayings of what is the case, rewriting "ordinary" sentences in a notation to reveal "necessary" relations amongst them, even those involving "material" concepts (e.g., empirical laws).

In the imagined final analysis, each elementary sentence, being logically independent of all the others (as in truth-functional logic), would be capable of being affirmed or denied, rightly or wrongly (each is true or false). A particular affirmation of "everything that is the case" would affirm/deny each elementary sentence in a manner consistent with the totality of its consequences, accomplishing in action a truth-functional assignment of T's and F's to each elementary sentence. Wittgenstein's point is that same-saying by affirmation and/or denial is definite and unambiguous.

Wittgenstein's Operator $N$ yields a step-by-step, formal construction of all possible sentences. The totality of the results of $N$'s constructive applications unify the formal aspects of logic and language in what Wittgenstein calls the "general form of sentence" (hereafter GFS). The TLP's notational devices are all devoted to making this rewriting project seem not only plausible but necessary.

Sentences are composed of symbols exhibiting dimensions of generalization on which we can formally "operate" in a step-by-step manner: aspects. If I think, for example, that the object $a$ is red (suppose $a$ and "red" appear in the final analysis), then a logical aspect of the thought is expressed with the sentential variable "$x$ is red." A general form, "$\phi(x)$," ranges over all elementary sentences sharing this form.

"Simple objects" are the ultimate nodes of inference in the final analysis. The forms of their possible configurations mirror all possible "internal" – that is, logical – relations among structures of thought, the common "form" and "substance" of all ways of thinking or imagining what might be so. An "internal" relation among thoughts $p$ and $q$ (or objects) is necessary in that they cannot fail to have and remain the thoughts (objects) they are: that is, $p$ is logically equivalent to $q$. Crystallizing the totality of both form and content, undefinable by further analysis or description, the names of simples are like coordinates and dimensions constituting "logical space." Philosophy (logic) is to construct "a system of signs of a definite number of dimensions – of a definite mathematical multiplicity" (5.475).

The final analysis would fully disambiguate language: different names indicate different objects. Thus the identity sign is eliminated from the proposed notation as redundant (5.53f.), with the cardinality of the universe expressed by the number of names – obviating the need for an *Axiom* of Infinity. Russell's second-order definition of identity as indiscernibility is unnecessary (5.5302ff.), his theory of descriptions reduced to configurations of names. Frege's treatment of sense and reference for names vanishes (6.232).

The right "mathematical multiplicity" would emerge a posteriori, as a result of research (5.55ff.). It is not part of the task of analysis to anticipate the actual compositions of sentential structures needed to represent reality, for example, the number of *n*-place relations or the extent of higher-order analysis or geometry required in physics.

Yet Wittgenstein urges that a complete a priori analysis of all the pure possibilities and mathematical multiplicities *must* be possible. We are capable of setting aside any factual truths, laws, or happenstances by seeing *through* our everyday activities of picturing facts *to* what is the case *if* a thought or sentence is true. Sentences are themselves facts, structures, none of which is a priori true, each of which may be compared with the facts. Form is the *possibility* of structure, the being-capable-of-being-true-or-false, or "sense" (*Sinn*) of a sentence (2.033).

This transposes the traditional dialectic between Realism and Idealism ("Skepticism") into a purely logical key. Analysis presents the truth, the whole truth, and nothing but the truth, *simply* – that is, determinately and clearly (3.23, 5.4541, 5.5563). It is coherent to imagine that all our affirmations of what is the case are false, that they are all true, or that some are true, some are false. Logic and mathematics run through all possibilities of thought. This is a move reminiscent of Kant, except that while Wittgenstein remarks that logic is "transcendental," he also holds the un-Kantian view that all logic is general (formal) logic, which is "analytic" and therefore "tautological."[19]

Like Kant, Wittgenstein never regarded mathematical knowledge as tautological. Aspects are his logicist-inspired way of articulating the "syntheticity" of thought at which Kant aimed. The Vienna Circle did not follow the philosophy of mathematics of the TLP, for they assumed that the logicist reduction of mathematics to logic was successful. Carnap, who attempted to reconstruct arithmetic and analysis as tautological and failed, relabeled the truths of logic and mathematics "analytic" but *not* "tautological," reserving the latter term, as we now do, for the smaller class of truth-functional tautologies.

19 Dreben and Floyd, 1991.