1 Introduction

One of the central problems within contemporary philosophy of mathematics is BENACERRAF'S DILEMMA, raised by Paul Benacerraf (1973).[1] In effect, Benacerraf charges that we cannot have both an empirically respectable semantics for number words and an empirically respectable epistemology of arithmetic. More exactly, Benacerraf argues that because (1a,b) have the same underlying "logical form," or semantic representation, given in (1c), '17' in (1b) is a NUMERAL (a name of a number), similar to 'New York.'

(1) a. There are at least three large cities older than New York.

b. There are at least three perfect numbers greater than 17.

c. There are at least three *F*s which bear *R* to *a*.

If so, then because (1b) is true, that numeral requires a referent. Moreover, the most obvious candidate referent is a *number*, which is presumably an abstract object. Hence, making semantic sense of basic arithmetic statements requires positing an ontology of abstract objects.

On the other hand, Benacerraf requires a CAUSAL THEORY of epistemology, whereby a knowledge report of the form '*S* knows that *p*' is true only if there is a causal chain linking *S* and what that knowledge is about. In the case of (1a,b), for instance, a causal theory would require a causal chain linking us to New York and the number seventeen, respectively. However, the latter is inconsistent with numbers being abstract: since abstract objects exist outside of space and time, they are causally inefficacious. So, if knowledge of the kind expressed by (1b) requires a causal link, then it would appear that arithmetic knowledge is generally impossible.

The traditional response to Benacerraf's Dilemma has been to reject the epistemic horn: even if arithmetic statements like (1b) refer to abstract objects, knowing what is expressed by such statements does not require a causal link between agents and those objects.[2] However, an interesting alternative strategy has emerged in recent times. According to it, we should question the primary assumption driving Benacerraf's semantic horn, namely that '17' in (1b) is a referring term. If it is instead a *non-referential* expression, then explaining the truth of (1b) may not require postulating numbers.[3]

[1] Not to be confused with another famous dilemma posed by Benacerraf (1965), known as Benacerraf's Identification Problem.
[2] See e.g. Shapiro (1997).
[3] See e.g. Nutting (2018).

This raises an important question. Apart from the vague intuition that (1a,b) have the same "logical form," why think that number words *ever* function referentially within true mathematical statements? Bob Hale (1987) offers one influential argument, the first premise of which is:[4]

> **Successful Reference**: If a range of expressions functions as singular terms in true statements, then there are objects denoted by expressions belonging to that range.

By definition, singular terms are expressions whose express semantic function is to refer, presumably to objects. Prototypical examples include names like 'Mars' in (2a) and definites like 'the planet Mars' in (2b).

(2) a. Mars is a red planet.
b. The planet Mars is red.

Now compare (2a,b) with (3a,b).

(3) a. Two is an even number.
b. The number two is even.

Given the surface syntactic similarities between (2a,b) and (3a,b), it would thus appear difficult to deny that the underlined expressions in (3a,b) are singular terms if those in (2a,b) are singular terms.

Hence the motivation for Hale's second premise:

> **Singular Terms**: Numerals, and many other numerical expressions besides, function as singular terms in many true statements (of both pure and applied mathematics).

By statements of "pure mathematics," Hale presumably has in mind those characteristic of a particular branch of mathematics, such as number theory. These may include formal statements of the theory, or their informal natural language counterparts. For example, (4a,b) might count as "pure" statements of number theory.

(4) a. $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}.\, S(x) = y$
b. Every natural number has a successor.

(3a,b) might also count as "pure statements," since they report something provable within number theory. In contrast, (5a,b) would presumably count as statements of "applied mathematics," as they report something involving a potential *application* of numbers, namely counting.

[4] Successful Reference, along with Singular Terms, Candidacy, and Existence below, are near direct quotations from Hale (1987).

(5) a. Mars has exactly two moons.
b. The number of Mars's moons is two.

Note that (5a,b) are not only plausibly equivalent, but also that 'two' seemingly functions as a singular term in (5b). Thus, not just statements of "pure arithmetic" are potentially relevant to Singular Terms.

Hale's final premise is:

Candidacy: Numbers are the best candidates to play the role of the referents of numerical expressions used referentially.

Intuitively, basic arithmetic is *about* numbers. Thus, insofar as examples like (3a,b) and (5b) are statements of pure and applied mathematics, broadly construed, they too are presumably *about* numbers. Furthermore, it is hard to see how (3a) could be true without Candidacy. Indeed, if 'two' referred to something other than a number, then (3a) would be false, presumably.

Jointly, Hale's premises imply:

Existence: Therefore, there exist objects denoted by numerical expressions, namely numbers.

Suppose (3a,b) are true. Then, by Successful Reference and Singular Terms, there is an object referenced by 'two' and 'the number two.' By Candidacy, that object is a number. So, by Existence, there exists an object which is the number two. This is just a restatement of REALISM, or the view that numbers exist. Accordingly, call this HALE'S ARGUMENT FOR REALISM.

To clarify terms, let's call numerical statements in which number expressions appear to function as singular terms referring to numbers, such as (1b), (3a,b), and (5b), APPARENTLY REFERENTIAL NUMERICAL STATEMENTS (ARNSs), and apparent singular terms featuring in those statements APPARENT SINGULAR TERMS (ASTs). [By NUMERICAL STATEMENTS, I mean utterances of sentences purporting to be about or otherwise feature number. And by NUMBER EXPRESSIONS, I mean simple or complex expressions purporting to be about or otherwise feature number, including 'two' in (3a) and (5a), 'the number two,' and 'the number of Mars's moons.'] Hale's Argument highlights two questions vital to the relationship between semantics and the ontology of number. First, are ARNSs true? Secondly, are ASTs featuring in ARNSs actually singular terms?

Three influential views within the philosophy of mathematics emerge depending on how these questions are answered. According to what I call REFERENTIALISM, ARNSs are true, and ASTs are singular terms. Versions of referentialism have been defended by e.g. Frege (1884), Shapiro (1997), and Hale and Wright (2001). According to what I call NON-REFERENTIALISM, ARNSs are true, but ASTs are not actually singular terms. Though various philosophers have proposed analyses suggesting non-referentialism, it has arguably been defended most forcefully and extensively by Hofweber (2005, 2007, 2016). Finally, according to what I call ERROR THEORY, although ASTs are singular terms, ARNSs are not actually true. Versions of error theory have been defended by e.g. Field (1980) and Leng (2005).

Clearly, if referentialism is correct, then so is Singular Terms. Thus, referentialism supports realism through Hale's Argument. In contrast, non-referentialism and error theory are both consistent with NOMINALISM—the view that numbers do not exist—though for importantly different reasons. This plausibly reflects a difference in philosophical methodology: whereas referentialists and non-referentialists agree that the question of whether numbers exist is to be determined largely, if not primarily, by the best available linguistic evidence, error theorists typically presuppose nominalism, for non-linguistic reasons. Thus, the dispute between referentialists and non-referentialists concerns the linguistic facts: Do they, or don't they, reveal ASTs to be genuinely referential? In contrast, error theorists typically take surface syntactic appearances at face value: ASTs really are referential. However, since their would-be referents do not exist, ARNSs are not true.

This short monograph is principally concerned with the referentialism/non-referentialism debate. The primary question to be addressed is whether, given the best available linguistic evidence, ASTs are singular terms referring to numbers. The connection to Benacerraf's Dilemma is immediate: if referentialism is correct, then the truth of ARNSs would seemingly straightforwardly vindicate realism, and with it Benacerraf's semantic horn. But if non-referentialism is correct, then not only is the truth of ARNSs consistent with nominalism, we would appear to have a novel and sufficiently general way out of Benacerraf's Dilemma.

Although my principal target here is the empirical motivation for referentialism and non-referentialism, this should also be of significant interest to error theorists. Even if the primary motivations for adopting nominalism are non-linguistic, whether ASTs are actually referential is an empirical matter. In this respect, one crucial aspect of error theory depends directly on the empirical viability of referentialism. Furthermore, if non-referentialism is correct, then there is no apparent need to claim that ARNSs are not true, contrary to intuition and basic mathematics. In this respect, non-referentialism would appear preferable to error theory, and so its other aspect plausibly depends indirectly on the empirical viability of non-referentialism.

Ultimately, I will argue here that the semantic evidence strongly supports referentialism, but for reasons which have been largely unappreciated within the philosophical literature. There are two notable tendencies among philosophers when discussing the relationship between numerical discourse and ontology. The first is to focus on a small handful of "mathematical" examples. Often, these include just statements of "pure mathematics," such as '2 + 2 = 4' or (4a,b). When other kinds of examples are discussed, these are often limited to statements of "applied mathematics" like (5a,b). The second tendency is to assume that number words function exclusively either as singular terms, or else as non-referential expressions. Thus, in cases like (5a,b), where 'two' appears to function *both* referentially and non-referentially, surface syntactic appearances must be misleading.

I believe that both tendencies are mistaken, and that recognizing this is crucial to understanding the ontological ramifications of numerical discourse. First, number words have *many* kinds of uses beyond those discussed above. For example, 'two' can be used not only as a numeral *and* as a quantificational expression, as witnessed in (5a,b), but also as a predicate and a modifier. Furthermore, it can be used to convey different kinds of numerical information—cardinal, arithmetic, ordinal, and measurement—each corresponding to different potential applications of numbers—counting, calculating, ordering, and measuring. Ultimately, an empirically adequate semantics for number words should not only explain how number words can be used in these various ways, but also, and perhaps more importantly, how the different meanings expressed by these various uses are *related*. And a semantics for number words can inform the question of whether numbers exist, presumably, only if it is empirically adequate.

My contention is that the only kind of semantic analysis capable of explaining these features overwhelmingly supports referentialism, and thus realism. More exactly, such an analysis will not only recognize that number words can take on a wide variety of different semantic functions and corresponding meanings, but also that these are related in virtue of sharing a certain element in common, namely a *number*. This in turn provides novel, empirical support for Singular Terms and Candidacy, suitably understood, thus resulting in a strengthened form of Hale's Argument. A philosophically significant, and perhaps surprising, consequence will be that *all* numerical discourse presupposes an ontology of numbers. This includes not just cases like (3a,b), which involve explicit reference to numbers, but also those like (5a), which do not. As a result, if numbers do not exist, then *none* of our numerical discourse is true—not just overtly "mathematical" statements, like (1b) or (3a,b), but even more mundane statements like (5a).

The rest of the monograph is divided into two sections. Section 2 outlines the empirical motivations for two influential strategies within the philosophical literature for dealing with examples like (5a,b), namely the substantival strategy and the adjectival strategy. Both promote or else are a form of referentialism or non-referentialism, and both can be deployed to address a particular instance of Hale's Argument, namely the Easy Argument for Numbers. I will argue that extant versions of both strategies face significant theoretical and empirical challenges.

In Section 3, I argue that both strategies are mistaken in virtue of wrongly assuming that number words featuring in examples like (5a,b) are exclusively referential or exclusively non-referential. After observing that number words have a wide variety of both referential and non-referential uses, I sketch a comprehensive semantics which not only provides meanings appropriate for all of these uses, but which also explains how those meanings are systematically related. I then explain how the resulting semantics affords a strengthened version of Hale's Argument, thereby undermining non-referentialism while also rendering error theory highly implausible.

Although this monograph is principally targeted at an ontological debate within the philosophy of mathematics, I hope that it may be potentially valuable to other areas of philosophy, and perhaps even linguistic semantics. There are analogous debates in other areas of philosophy, notably with respect to proper names and natural kind terms, and I think certain methodological points raised here carry over to these other areas. And while the semantics developed is by no means revolutionary, it does raise important empirical questions which should be of independent linguistic interest.

That said, there are numerous fascinating theoretical and empirical issues I have needed to ignore, for space. These include a more complete discussion of nominalist programs within the philosophy of mathematics, possible rejoinders to my arguments, empirical matters relevant to deciding the lexical meanings of number words, and, perhaps most obviously, how best to deal with Benacerraf's Dilemma in light of my arguments for referentialism. I intend to return to these matters in future work.

2 Substantivalism and Adjectivalism

As mentioned, this monograph is principally concerned with a recent debate regarding the meanings of number expressions and their ontological import. Specifically, it centers on Hale's Argument. The claim is that if the underlined expressions in (6a-c) are singular terms referring to numbers, then the truth of (6a-c) implies that numbers exist.

(6) a. <u>Four</u> is an even number.
b. <u>The number four</u> is even.
c. <u>The number of Jupiter's moons</u> is <u>four</u>.

But what exactly *is* a singular term, anyway, and why should we think that the underlined expressions in (6a-c) are among them? To this end, one might follow Dummett (1973) in characterizing singular terms via the kinds of inferences they (and only they) license. For example, whereas prototypical singular terms, such as names, apparently license the intuitive entailments in (7), prototypical non-referential expressions, such as 'nobody' or 'most Americans,' apparently do not.

(7) a. Mary is at least 18 years old or younger ⊨ Mary is at least 18 years old or Mary is younger than 18
   b. Mary is intelligent ⊨ Someone is intelligent
   c. Mary is female and Mary supports legalizing marijuana ⊨ Someone is female and supports legalizing marijuana

Note that numerals and definites like 'the number four' pattern similarly to names in this respect.

(8) a. {Four/The number four} is greater than or equal to five or less than five ⊨ {Four/The number four} is greater than or equal to five or {four/the number four} is less than five
   b. {Four/The number four} is even ⊨ Some number is even
   c. {Two/The number two} is even and {two/the number two} is prime ⊨ Some number is even and prime

Something similar can be said for (6c), it seems.

(9) The number of Jupiter's moons is four ⊨ The number of Jupiter's moons is something, namely four

So, if licensing these entailments suffices for singular termhood,[5] examples like (6a-c) would seemingly vindicate Singular Terms.

This section is organized around the apparent entailment in (9), which has received quite a lot of recent philosophical attention. In fact, (9) underwrites a certain contentious argument for realism, known as THE EASY ARGUMENT FOR NUMBERS. Intuitively, (10) entails (6c).

(10) Jupiter has four moons.

If so, and if (9) is a genuine entailment, where 'something' ranges over numbers,[6] then realism seemingly follows in virtue of successfully counting Jupiter's moons. As we will see, however, there is nothing approaching philosophical consensus as to whether (9) is a genuine entailment.

5 I will return to this assumption in §3.3
6 See Moltmann (2008) for relevant discussion.

Although I have framed this section around the Easy Argument, due to its prominence within the literature, it is important to appreciate its role within the context of Hale's Argument, from the outset. After all, apparent identity statements like (6c) are just one instance of number expressions purporting to function as singular terms within true numerical statements. Thus, even if the apparent entailment in (9) is not genuine, as some philosophers maintain, this alone would not undermine Singular Terms, given other examples like (6a,b). Rather, Singular Terms is false only if *no* terms purporting to refer to numbers are genuinely referential.

Put differently, whereas some philosophers have recently argued that the best available semantic evidence suggests rejecting the Easy Argument, this alone will not establish *non-referentialism*, consistent with nominalism. Conversely, if the only criteria for qualifying as a singular term is that an expression licenses entailments like (7-c), and yet the best available semantic evidence requires rejecting (9), then the Easy Argument would provide no support for realism. In this respect, the Easy Argument nicely illustrates the broader ontological question this monograph intends to address.

The rest of the section is laid out as follows. §2.1 discusses Frege (1884)'s observation that number expressions appear to have both referential and non-referential uses, and how this gives rise to the present philosophical debate. Specifically, it gives rise to two popular philosophical strategies—the substantival strategy and the adjectival strategy. The rest of the section then spells out the motivations for, and challenges facing, versions of these two strategies. Specifically, §2.2 discusses a version of the substantival strategy, suggested by Crispin Wright (1983), and the linguistic challenges for his neo-logicist program. §2.3 discusses three versions of the adjectival strategy. As we will see, while only one version purports to establish non-referentialism, all three face significant empirical challenges.

2.1 Two Strategies of Analysis

Gotlob Frege (1884, §57) notes that number words appear to have different uses in natural language. Specifically, they appear to have referential uses, as witnessed in (11b), and non-referential uses, as witnessed in (11a).

(11) a. Jupiter has four moons.
b. The number of Jupiter's moons is four.

One of Frege's major contentions is that while statements like (11a) appear to be about objects – e.g. Jupiter – they are really a statement about concepts – e.g. being a moon of Jupiter:

> To throw light on the matter, it will help us to consider number in the context of a judgment that brings out its ordinary use. If, in looking at the same external phenomenon, I can say with equal truth 'This is a copse' and 'These are five trees,' or 'Here are four companies,' then what changes here is neither the individual nor the whole, the aggregate, but rather my terminology. But that is only a sign of the replacement of one concept by another. This suggests . . . that a statement of number contains an assertion about a concept. . . If I say 'The King's carriage is drawn by four horses,' then I am ascribing the number four to the concept 'horse that draws the King's carriage.' (§46)

In modern logical notation, Frege contention can be rendered as (12), translating (11a), where 'M' abbreviates 'is a moon of Jupiter.'

(12) $\exists x_1, \ldots, x_4. [M(x_1) \wedge \ldots \wedge M(x_4) \wedge x_1 \neq x_2 \wedge \ldots \wedge x_3 \neq x_4] \wedge [\forall z. M(z) \rightarrow z = x_1 \vee \ldots \vee z = x_4]$

The first conjunct asserts that at least four things are moons of Jupiter, while the second asserts that there are no other such moons. Together, then, (11a) will be true if Jupiter has exactly four moons.[7] On this analysis, 'four' is analyzed as a second-order property of concepts, namely those whose extensions consists of exactly four objects. Notice that (12) could be true in a model containing only five objects: four moons and Jupiter. Hence, the truth of (12) is consistent with nominalism.

Despite this, Frege goes on to observe that it is always possible to paraphrase apparently non-referential uses like (11a) in terms of apparent identity statements like (11b).

> Since what concerns us here is to define a concept of number that is useful for science, we should not be put off by the attributive form in which number also appears in our everyday use of language. This can always be avoided. For example, the proposition 'Jupiter has four moons' can be converted into 'The number of Jupiter's moons is four.' Here the 'is' should not be taken as the mere copula … Here 'is' has the sense of 'is equal to,' 'is the same as.' We thus have an equation that asserts that the expression 'The number of Jupiter's moons' designates the same object as the word 'four.' (§57)

Frege was primarily interested in developing an ideal logical language suitable for science: a *Begriffsschrift*. Within such a language, number expressions would always function referentially. Thus, Frege's suggestion is that non-referential uses like (11a) should be paraphrased as (11b), where the latter is an identity statement equating two numbers: the number of Jupiter's moons

[7] According to NASA (`https://solarsystem.nasa.gov/moons/jupiter-moons/overview`), Jupiter may have up to seventy-nine moons. Nevertheless, I will continue using Frege's examples due their influence, hoping no substantial confusion results.

and the number four. Formally, (11b) can be analyzed as (13), where '#' is a cardinality-function mapping a concept to a natural number (finite cardinal) representing how many objects fall under that concept.

(13) $\#[\lambda x.\, M(x)] = 4$

As an identity statement, (13) references a number directly. Thus, unlike (12), (13) would be true only if numbers exist. It would not be true in a model containing just four moons and Jupiter, for instance. So, whereas the truth of (12) is consistent with nominalism, (13) implies realism.

In sum, not only do number expressions appear to serve importantly different semantic functions, these different uses appear to have significantly different ontological consequences. Accordingly, Dummett (1991, p. 91) gives labels to two positions one might take in light of Frege's observation:

> Number-words occur in two forms: as adjectives, as in ascriptions of number, and as nouns, as in most number-theoretic propositions. When they function as nouns, they are singular terms, not admitting of the plural; Frege tacitly assumes that any sentence in which they occur as adjectives may be transformed either into an ascription of number . . . or into a more complex sentence containing an ascription of number as a constituent part. Plainly, any analysis must display the connection between these two uses. . . Evidently, there are two strategies. We may first explain the adjectival use of number-words, and then explain the corresponding numerical terms by reference to it: this we may call *the adjectival strategy*. Or, conversely, we may explain the use of numerals as singular terms, and then explain the corresponding number-adjectives by reference to it; this we may call *the substantival strategy*.

Thus, according to THE SUBSTANTIVAL STRATEGY, apparently non-referential uses like (11a) are to be analyzed in terms of referential uses like (11b). In other words, (11a,b) both receive something like the ontologically committing truth-conditions in (13), following Frege. In contrast, according to THE ADJECTIVAL STRATEGY, apparently referential uses like (11b) are to be analyzed in terms of non-referential uses like (11a). Thus, both (11a,b) receive something like the ontologically non-committing truth-conditions in (12).

Obviously, Frege advocated the substantival strategy, at least for the purposes of developing an ideal logical language. However, despite Frege's considerable influence, we will see that the adjectival strategy has gained increasing popularity among philosophers in more recent times.

2.1.1 The Easy Argument for Numbers

Whether numbers exist is an important, longstanding philosophical question. Thus, it would be surprising if it could be answered simply by peering through a