# 1

# Introduction

## Mathematics and the World of Experience

> Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But it cannot be understood unless one first learns to comprehend the language and read the letters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth. . ..
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> —Galileo, *Il Saggiatore*, 1623[1]

### 1.1 Kant and the Theory of Magnitudes

My aim in writing this book is to transform our current understanding of Kant's philosophy of mathematics, and in doing so, our understanding of Kant's account of the world of experience. Mathematics and the world are more intimately intertwined in Kant's philosophy than many have appreciated. I will argue that in Kant's account, mathematics is a science of magnitudes, and the world of experience is a world of magnitudes. That is, Kant's philosophy of mathematics, pure as well as applied, is grounded in a theory of magnitudes; at the same time, all objects of experience are and all their real properties have magnitudes, so that the world we experience is a world of magnitudes. The world is fundamentally mathematical in character, and in taking magnitudes as its object of study, pure mathematics is about the world. This is particularly true of geometry – the science of continuous spatial magnitudes – which in Kant's time still enjoyed a certain pride of place in thinking about mathematics and in an understanding of the mathematical character of the world.

This reorientation in Kant's account of mathematics and his account of experience has important consequences for our understanding of both mathematical and theoretical cognition. The role of intuition in both is a major

1 Translated in Popkin (1966, p. 65).

theme of this book. According to Kant, a magnitude is a homogeneous manifold in intuition. I will argue that in Kant's view representing magnitudes as magnitudes at all depends on intuition, because intuition allows us to represent a homogeneous manifold. The fact that intuition allows us to represent a homogeneous manifold has not been appreciated, yet it has important implications for Kant's claim that mathematical cognition and all human cognition depend on sensible intuition. Moreover, I will argue that the singularity of intuition is best understood as a mode of representing singularly, one that is compatible with representing a homogeneous manifold in intuition.

Another closely related theme is that the role of intuition in Kant's account of mathematical cognition makes both our mathematical cognition and our cognition of the mathematical features of experience more concrete than we are apt to think today. To shift for the nonce to our contemporary parlance of particulars, the role of intuition allows us to represent relatively concrete particular magnitudes in space and time. By concrete, I mean that intuition allows us to represent spatial and temporal particulars; by *relatively* concrete, I here mean that the role of intuition in representing the mathematical features of those particulars does not include the representation of causal relations, a common feature attributed to concreta in our contemporary metaphysical theorizing about them.[2] The fact that intuition represents singularly is what allows us to represent objects *in concreto*, and hence to represent particular objects. Both pure mathematical cognition and our representation of objects of experience rests on the singular representation of concrete continuous and discrete homogeneous manifolds in intuition.

Kant's account, however, is both complex and nuanced. First, Kant allows for multiple roles for intuition in mathematical cognition, not just the representation of concrete homogeneous manifolds. For example, intuition plays an important role in allowing us to represent succession, which is required for arithmetical cognition. Second, mathematical cognition does not merely rest on the intuitive representation of particular concrete magnitudes; it also essentially depends on concepts, in particular, the categories of quantity, as well as rules for the representation of magnitudes, that is, schemata. Third, Kant's primary notion of magnitude is concrete, but he also makes room for a more abstract notion of magnitude. The contrast between concrete and more abstract representations of magnitude correspond to a distinction Kant draws between two sorts of magnitude, *quanta* and *quantitas*. Furthermore, this distinction is tied to Kant's obscure and complicated understanding of number.

[2] This is a provisional characterization of "relative concreteness." I will examine Kant's notion of concreteness in more depth in Chapter 5.

A full account of Kant's philosophy of mathematics would require sorting out all the nuances and complexities of Kant's theory of magnitudes and bringing them to bear on what he says about geometry, arithmetic, algebra, and analysis. I cannot hope to do all of that between the covers of one book. Instead, this work will focus on the foundations of Kant's theory of magnitudes and its relation to both Kant's account of mathematical cognition and our cognition of objects of experience. I plan to follow with another book that will delve more deeply into the implications of the theory of magnitudes in Kant's account of geometrical, arithmetic, and algebraic cognition, as well as in the foundations of analysis. The present book, I hope, will speak to those readers interested more broadly in the foundations of Kant's theoretical philosophy, with an eye to his philosophy of mathematics and mathematical cognition and its implications for Kant's account of experience.

There are, however, a few features of the foundations of Kant's theory of magnitudes that cannot be fully and satisfactorily addressed until the details of his account of mathematical cognition have been explained. Those include aspects of the distinction between two notions of magnitude, *quanta* and *quantitas*, and their relation to number. I cannot give a complete account of number in this book, but I devote several sections to the distinction between *quanta* and *quantitas*, and discuss Kant's understanding of number and its relation to the Greek mathematical tradition.[3] We will see that in Euclid and the Euclidean tradition, the understanding of continuous magnitude is entangled with that of number in several ways. Those entanglements are also found in Kant, but unraveling them requires an in-depth focus on Kant's arithmetic. There are therefore a few claims about *quanta* and *quantitas* and their relation to number whose full defense depends on promissory notes to be redeemed in the second book.

I will argue for an interpretation of the foundation of Kant's theory of magnitudes and its relation to his understanding of experience based both on a close reading of the texts and on placing those texts in historical context. My aim is to determine Kant's views as accurately as I can without attempting to evaluate Kant's views from our contemporary perspective. It will be a sufficient accomplishment to get Kant's views right. I will press, however, to the limits of Kant's theorizing about magnitudes, that is, to the limits of how much he was able to develop and articulate his views given the time he had to devote to the topic. I will also move beyond what Kant explicitly says in order to reconstruct the key assumptions underlying his theory and to determine his views with regard to those assumptions.

[3] For an argument that Kant's conception of number includes both cardinal and ordinal elements, see Sutherland (2017). Future work will more fully address the relationship of Kant's conception of number to *quanta* and *quantitas* and also explain Kant's understanding of irrational numbers.

This work is meant to be generally accessible to philosophers interested in Kant's account of experience, and will not presuppose anything but a rudimentary understanding of mathematics nor a familiarity with the history of mathematics. From those steeped in philosophy of mathematics or its history, I beg patience, and hope that there is ample material of interest to hold their attention. In the remainder of this introductory chapter, I say a bit more about the themes mentioned above to orient the reader, before closing with an overview of the book.

## 1.2 Mathematics Then and Now

Kant's view of mathematics as a science of magnitudes was common in the eighteenth century, but it is strikingly different from our contemporary way of thinking. This is not the place to recount the history and philosophy of mathematics from Kant to the present, but there are two ways in which Kant's views are different that are important to highlight. The first is that mathematics has become a science of number rather than magnitudes. The "arithmetization" of mathematics over the course of the nineteenth century placed natural number firmly at the foundation of mathematics, encouraged a more abstract understanding of number, and introduced a separation between mathematics and the world. Pure mathematics is no longer about the world insofar as it is constituted by magnitudes. Instead, natural numbers are used to construct the rationals, the reals, and complex numbers, and once these foundations are complete, the relation between mathematics and the world can be taken up as an issue of applied mathematics. Further work at the end of the nineteenth century on the foundations of arithmetic itself attempted to provide a foundation of number in terms of notions more basic than natural numbers and their arithmetic. These notions were supplied by logic and the emerging theory of sets, both of which were thought of in abstract terms, which reinforced a more abstract understanding of number. At the same time, the development of axiomatics solidified the growing primacy of arithmetic over geometry, as well as a separation of pure mathematics from the world to which it was applied. "Pure" geometry came to be viewed as the study of the consequences of various sets of axioms apart from whether those axioms describe physical space. As a result of all these developments, pure mathematics shifted in the nineteenth century from being a science of magnitude to being first and foremost a science of number.[4]

The fact that the arithmetization of mathematics led to a more abstract understanding of mathematics and a separation between pure mathematics

[4] According to Petri and Schappacher (2007), the view of mathematics as a science of magnitudes was not extinguished until 1872.

and the world is closely related to a further issue: the drive to emancipate mathematics from intuition. The eighteenth century saw remarkable advances in mathematics, especially in analysis, which included what we now call calculus. Nevertheless, when it came to foundational questions, mathematicians and philosophers still reverted to thinking of mathematics as a science of magnitudes, and many of those mathematicians and philosophers particularly concerned with foundations thought that intuition, in particular, geometrical representations, played an important role in securing the meaning and certainty of the most basic concepts and propositions of mathematics. Kant was among them.[5]

Kant radically departs from previous philosophers in elevating the status and role of intuition in all human cognition. Previous philosophers distinguished between what we receive through the senses from what we represent through the intellect, and addressed how they are related. Kant argues for a deeper difference, arguing that that intuitions are a fundamentally distinct kind of representation from concepts and belong to their own faculty. Moreover, intuitions are representations in the pure forms of space and time, which allow us to represent spatial and temporal features of the world, a role which had traditionally been assigned to empirical perception. Kant argued that space and time were forms of intuition, and hence that intuitions could be not just empirical but pure and *a priori*. Kant also departs from his predecessors in holding that intuition is required for all theoretical cognition, and in particular that pure intuition is required for all mathematical cognition. Kant relies on his distinction between analytic and synthetic propositions, that is, those propositions whose truth is grounded in the content of concepts and the containment relations among them, and those propositions whose truth is not. Kant claims that mathematical propositions are synthetic, and hence require intuition, and in particular pure intuition, to ground them. Kant met almost immediate resistance from philosophers in the continental rationalist tradition following Leibniz who rejected the claim that theoretical cognition, including mathematical cognition, depends on a nonconceptual form of representation. Even some of his allies were troubled and challenged him. They thought that geometry might plausibly depend on a pure intuition of space, but it is less obvious that arithmetic and algebra depend in any way on intuition. Nevertheless, Kant's critical philosophy and his claim about the role of intuition in mathematical cognition gained wide influence.

The nineteenth-century arithmetization of mathematics and foundations of arithmetic arose against this backdrop. Mathematicians answered the call for rigor to address problems in the foundations of analysis by rejecting any

5 This is a rather rough summary of a complex history. See Sutherland (2020b) for a more detailed account of Kant's relation to the history of analysis.

appeal to intuition, geometrical or otherwise. Far from helping establish certainty, intuition came be seen as not just unreliable but potentially misleading. The arithmetization of mathematics meant that more was at stake in Kant's claim that intuition is required for arithmetic in particular. At the same time, the rejection of intuition in arithmetic put great pressure on that claim. Frege's development of logic extended what could be expressed and derived within it, allowing Frege to expand and shift the notion of analyticity and claim that arithmetic is in fact analytic and does not depend for its justification on intuition. Russell stated that "formal logic was, in Kant's day, in a very much more backward state than at present," and that properly understood, mathematical reasoning "requires no extra-logical element" (Russell (1903), p. 457). He held that advances in both logic and mathematics itself eliminate the need for intuition. The greater logical resources that could be brought to bear and the drive to eliminate intuition led Russell to be quite dismissive of Kant's philosophy of mathematics.[6]

Not all agreed with the banishment of intuition, however. Hilbert, Poincare, and Brouwer each at some point and in their own way defended the idea that intuition has more than a heuristic role to play in our knowledge of mathematics. Some of these defenders looked to Kant for inspiration, even when their understanding of the role of intuition differed from his. What is striking from a historical perspective is that, whether one agreed or disagreed with Kant, the terms of the debate were set by him. But if we are to truly understand Kant's philosophy of mathematics, we will have to reconstruct a view of mathematics prior to its arithmetization, and that requires comprehending as best we can the idea that mathematics is a science of magnitudes, as well as Kant's account of the role of intuition in representing magnitudes both in mathematics and in experience.

One of the primary aims of this book is to bring to life this older way of thinking about mathematics. But because our modern way of thinking is deeply embedded in higher mathematics and even shapes basic mathematics education, it is difficult to shed our presumptions when reading Kant's claims concerning mathematics. The best way to recover the earlier way of thinking is to return to its roots in Euclid and the Euclidean tradition following him. The influence of Euclid's *Elements* can hardly be overstated; it was the model of mathematical reasoning and a paradigm of scientific knowledge for more than two millennia and was responsible for the dominance of geometry over arithmetic during that time. As De Risi notes, there were hundreds of translations of and commentaries on the *Elements*, and its dissemination and

[6] See Friedman (1992), especially pp. 55–6, as well as Friedman (2013) for a sustained argument that we can still learn a great deal from understanding Kant's views of mathematics and natural science, despite – in fact with the aid of – advances in our understanding of logic, mathematics, and physics.

influence throughout Europe was "only matched by the Bible and by a few other writings of the Fathers of the Church."[7] Even those who aspired to replace rather than modify the *Elements* began their studies with it and reacted against it. But Euclid's *Elements* contains more than mere geometry. An essential component is a theory of ratios and proportions among magnitudes, a theory attributed to Eudoxus.[8] This crucial part of the *Elements* set the framework for thinking about magnitudes in the Euclidean tradition and persisted into the nineteenth century. The *Elements* also contains books on number and the basic properties of numbers, including propositions governing the ratios and proportions among them. The conception of number expounded there influenced the understanding of number for nearly two millennia. The long Euclidean tradition included important challenges and modifications to the *Elements*, and there were of course remarkable advances in mathematics, particularly from the beginning of the Renaissance and through the eighteenth century. Nevertheless, the framework for thinking about mathematics, and in particular for thinking about the foundations of mathematics and about mathematical cognition, was strongly influenced by the Euclidean tradition and the Euclidean theory of magnitudes. That framework was still dominant in the eighteenth century.

We will keep Euclid's *Elements* close at hand throughout this book in order to understand Kant's very different way of thinking about mathematics. I will point out ways in which the long Euclidean tradition diverged from Euclid and describe developments during and after the Renaissance when they are important for understanding Kant. Obviously, a history of mathematics from Euclid to the eighteenth century is well beyond the scope of this work and what I highlight is quite selective. After discussing Kant's views of mathematics and magnitudes and their relation to experience in Part I, I will give a relatively brief and focused presentation of key features of Euclid's *Elements* that shaped the understanding of mathematics into the eighteenth century. That will put us in a position to dive more deeply into Kant's understanding of mathematics and its relation to the world in Part II.

Recovering Kant's understanding of mathematics requires a shift not just in an understanding of foundations, but in their aim. During and after the arithmetization of mathematics, the goal of foundations was to resolve various problems in analysis and to explain the nature of numbers by giving an account of certain mathematical notions (real, rational, natural numbers), in terms of more basic notions (rational numbers, natural numbers, logical and set-theoretic notions, respectively), and to do so in a rigorous way that would

[7] De Risi (2016, p. 592).

[8] Euclid compiled previous works of mathematics in writing the *Elements*, and the basic content of parts of it was attributed to various authors, including Eudoxus, as will be discussed in more detail in Chapter 6.

ground inferences. The primary focus was on providing a foundation for mathematics itself. Kant's aims were quite different. First and foremost, Kant wished to provide an explanation of the possibility of mathematical cognition, which includes both basic judgments, such as the judgment that there are seven apples in the bowl on the table, as well as what is required for higher mathematics, pure and applied, such as the derivation of Newton's law of universal gravitation. Kant's aim is to provide an explanation of mathematical cognition in terms of our most basic cognitive capacities. Those elements are the categories and the pure forms of intuition, so that Kant's explanation of the possibility of mathematical cognition is grounded in them.

This is not to say, however, that Kant had first settled on his theory of the categories and pure intuition, even its general shape, before addressing the foundations of mathematical cognition. Indeed, Kant's reflections on mathematical cognition, particularly in the *Prize Essay* period in the years 1762–4, was a driving force in the development of his critical philosophy, including his conviction that there is a class of truths that cannot be reduced to logical relations among concepts and that we have pure forms of *a priori* intuition. The development of Kant's critical understanding of the categories and the pure forms of intuition was strongly influenced by his philosophy of mathematics, and it offers more insights than have been generally appreciated. This is a story worthy of its own monograph, but it is one we will have to largely set aside here.[9]

What is important for our present purposes is that Kant's primary aim with respect to mathematics was to provide a foundation of mathematical cognition rather than a foundation of mathematics in our modern sense. The two sorts of foundations are inextricably linked; nineteenth-century foundations were often motivated by epistemological concerns, and Kant's understanding of mathematical cognition is conditioned by his understanding of the nature of mathematics. There is no easy division between the two. Nevertheless, the difference in focus and emphasis between Kant and post–eighteenth-century approaches is significant. In Kant's account, we attain mathematical knowledge through our cognition of magnitudes, and hence the focus of his foundations is on explaining our ability to cognize magnitudes in both pure mathematics and in experience. This is not to say that one cannot learn a great deal about Kant's philosophy of mathematics and philosophy of science by foregrounding Kant's interaction with the mathematics and science of his day, while leaving Kant's account of our cognition of magnitudes in the background; indeed, a good deal of very good work in recent decades has done just

[9] For relatively recent work focusing specifically on Kant's philosophy of mathematics in and after the Prize Essay period, see especially Carson (1992), Rechter (2006), and R. L. Anderson (2015).

that. But the focus of the present work will be on Kant's account of the foundations of mathematical cognition as a cognition of magnitudes.

## 1.3 Mathematics in Kant's Theoretical Philosophy

As I have already indicated, understanding Kant's theory of magnitudes is not merely crucial for his philosophy of mathematics; it is important for his entire critical philosophy. Because it is more important than is often recognized, this claim is worth defending here at the outset, though it is the book as a whole that makes the case.

Kant's mature philosophy only emerged with the *Critique of Pure Reason* (henceforth *Critique*),[10] but Kant's early reflections on mathematical cognition in 1762–4 were a key factor in moving Kant toward his view of the role of intuition in human cognition. Kant's reflections on the possibility of demonstrating God's existence during this period were certainly important to his critical assessment of and emancipation from Leibnizian and Wolffian metaphysics, as is clear in his essay *The Only Possible Argument for the Existence of God*. It was, however, his investigation of mathematical cognition in *Inquiry Concerning the Distinctness of the Principles of Natural Theology of Morals* (henceforth either *Inquiry* or *Prize Essay*) and *Attempt to Introduce the Concept of Negative Magnitudes into Philosophy* (henceforth *Negative Magnitudes*) that convinced him that Leibnizian and Wolffian rationalism based solely on conceptual representation and the relations among concepts could not account for mathematical cognition, and moved him toward his understanding of pure intuition and its role in human cognition.[11] I primarily focus on Kant's views in the critical period, in which the influence of his philosophy of mathematics on his theoretical philosophy is readily apparent, at several different levels.

The first level of that influence is well known, but bears review. Kant states in the *Prolegomena to Any Future Metaphysics* (henceforth *Prolegomena*) that metaphysicians must answer the question, "How are synthetic *a priori* cognitions possible?" and that the whole of transcendental philosophy is the answer (4:278–9). Kant claims that if metaphysics were possible, it would rest on synthetic *a priori* cognitions, but that it is disputable whether there are any

[10] Despite the deep importance of the two other critiques to Kant's philosophy as a whole, I will usually refer to *The Critique of Pure Reason* simply as the *Critique*. Our focus will be primarily on the role of magnitude in Kant's account of theoretical cognition in the *Critique of Pure Reason*, save one relatively short excursion into the *Critique of Judgment*.

[11] See R. L. Anderson (2015) for a recent particularly lucid and helpful account of Kant's reaction to Leibnizian and Wolffian rationalism, with a focus on the development of Kant's understanding of the analytic-synthetic distinction starting in the pre-critical period. For a broader account of Kant's reaction to rationalism, see Hogan (2009).

such cognitions to support metaphysics at all. Nevertheless, Kant says, to motivate and justify the question of how synthetic *a priori* knowledge is possible, we need not first establish that it is possible, because it is actual: we have clear examples of synthetic *a priori* cognitions in pure mathematics and pure natural science (4:275). Kant claims that we can "confidently say" that pure mathematics and pure natural science contain *a priori* cognitions; he adds that their status is "uncontested," and that these examples are "plenty and indeed with indisputable certainty actually given" (4:276). This is what justifies the analytic method he says he employs in the *Prolegomena*, that is, starting from the fact that we have synthetic *a priori* knowledge and seeking an explanation for its possibility (4:279).

Although Kant claims to employ the synthetic method in the *Critique*, and so presumably does not start with the assumption that we have a specific sort of cognition and then seek its explanation, he makes claims similar to the *Prolegomena* about the cognitions of pure mathematics and pure natural science. In the B-Introduction, for example, he states that, in contrast to the status of the propositions of metaphysics, pure mathematics "certainly contains synthetic *a priori* propositions" (B20). He also adds that since mathematics and pure natural science "are actually given, it can appropriately be asked **how** they are possible; for that they must be possible is proved through their actuality" (B20).

It is important, of course, to distinguish between the claim that the propositions of pure mathematics and pure natural science are indisputably certain, and the claim that their status as synthetic *a priori* cognitions is indisputably certain. The certainty of 2 + 2 = 4 is not the same as the certainty that this proposition is synthetic *a priori*, and although Kant states that their status as synthetic *a priori* cognitions is uncontested, he gives arguments to support his claim. Even the *Prolegomena*, which he says employs the analytic method and hence assumes that we have synthetic *a priori* cognition, provides considerations in favor of this claim in the Preamble. His tone in both the *Prolegomena* and the *Critique*, however, suggests that little real argument is needed, only careful reflection in light of the proper characterizations of the *a priori*/*a posteriori* and analytic/synthetic distinctions. The considerations Kant brings to bear in the B-Introduction of the *Critique* borrow almost verbatim from the *Prolegomena* Preamble. Concerning the apriority of mathematical judgments, Kant treats it as sufficient to simply attend to marks of apriority. Kant argues that necessity and universality are sure criteria of *a priori* cognitions and that this makes it is easy to show that there are *a priori* judgments in human cognition: "one need only look at all the propositions of mathematics," and he seems to think that no more argument is required. He also states that in the proposition "every event has a cause," the concept of a cause "obviously" contains the concept of necessity, which he cites in support of his claim that the proposition is necessary (B4–5).