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Chapter 1
Hypothesis testing

- Understand the nature of a hypothesis test, the difference between one-tailed and two-tailed tests, and the terms null hypothesis, alternative hypothesis, significance level, rejection region (or critical region), acceptance region and test statistic.
- Formulate hypotheses and carry out a hypothesis test in the context of a single observation from a population which has a binomial distribution, using:
  - direct evaluation of probabilities
  - a normal approximation to the binomial.
- Interpret outcomes of hypothesis tests in context.
- Understand the terms Type I error and Type II error in relation to hypothesis tests.
- Calculate the probabilities of making Type I and Type II errors in specific situations involving tests based on a normal distribution or direct evaluation of binomial probabilities.

1.1 Introduction to hypothesis testing

WORKED EXAMPLE 1.1

In a test the questions are all multiple choice. Each question has five possible choices.

a In a test with twelve questions, one student gets four questions correct. Test, at the 10% significance level, the null hypothesis that the student is guessing the answers.

b In a further test there are 120 questions. The same student took the test and got 32 correct. Test, at the 10% significance level, whether there is evidence to show the student is guessing the answers.

Answer

a Let X be the number of correct answers achieved. — Define a random variable.

$X \sim B(12,\ 0.2)$ — State the distribution.

$H_0: p = 0.2,\ H_1: p > 0.2$ — State the null and alternative hypotheses.

$P(X \geqslant 4) = 1 - P(X \leqslant 3)$ — Calculate the test statistic.

$$= 1 - \left(\binom{12}{0}0.8^{12} + \binom{12}{1}0.2^{1}0.8^{11} + \binom{12}{2}0.2^{2}0.8^{10} + \binom{12}{3}0.2^{3}0.8^{9}\right)$$

$$= 1 - 0.795 = 0.205$$

$0.205 > 0.1$

Accept H_0. There is sufficient evidence to show that the student is guessing answers.

Compare the result with the significance level and write the conclusion in the context of the question.

b $X \sim B(120, 0.2) \approx N(24, 19.2)$

State the distribution.

$H_0 : p = 0.2$ or $\mu = 24$
$H_1 : p > 0.2$ or $\mu > 24$

The parameter for the hypotheses can be p or μ, but must be consistent.

$$\begin{aligned} P(X \geqslant 32) = P(X \geqslant 31.5) &= 1 - P(X \leqslant 31.5) \\ &= 1 - \Phi\left(\frac{31.5 - 24}{\sqrt{19.2}}\right) \\ &= 1 - \Phi(1.712) \\ &= 1 - 0.957 = 0.043 \end{aligned}$$

Remember to use a continuity correction when a binomial is approximated to a normal.

$0.043 < 0.1$

Reject H_0. There is evidence to suggest that the student might not be guessing answers.

Compare the result and write your conclusion in the context of the question.

EXERCISE 1A

1 Write down the null and alternative hypotheses for the following tests, defining the meaning of any parameters.

a i Daniel wants to test whether the proportion of children in his school who like football is higher than 60%.

ii Elsa wants to find out whether the proportion of households with a pet is higher than 1 in 3.

b i The proportion of faulty components produced by a machine was 6% and the manager wants to check whether this has decreased following a routine maintenance check.

ii Joseph thinks that fewer than half of all children eat 5 or more pieces of fruit a day and wants to confirm this by using a hypothesis test.

2 In each question, you are given null and alternative hypothesis (where p stands for the population proportion), the significance level and the observed data. Decide whether or not there is sufficient evidence to reject the null hypothesis.

a i $H_0 : p = 0.3,\ H_1 : p > 0.3$, significance level 5%, observed 15 successes out of 40 trials

ii $H_0 : p = 0.6,\ H_1 : p > 0.6$, significance level 10%, observed 23 successes out of 45 trials

b i $H_0 : p = 0.5,\ H_1 : p < 0.5$, significance level 10%, observed 18 successes out of 40 trials

ii $H_0 : p = 0.45,\ H_1 : p < 0.45$, significance level 3%, observed 23 successes out of 60 trials.

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PS **3** A national opinion poll claims that 40% of the electorate would vote for Party R if there were an election tomorrow. A student at a large college suspects that the proportion of young people who would vote for them is lower. She asks 16 fellow students, chosen at random from the college roll, which party they would vote for. Three choose Party R. Show that, at the 10% significance level, there is evidence that the reported figure is too high for the young people at the student's college.

PS **4** The lengths of nails produced by a machine have a normal distribution with mean 2.5 cm. A random sample of 16 nails is selected from a drum containing a large number of these nails. The nails are measured and 13 are found to have length greater than 2.5 cm.

a Test, at the $2\frac{1}{2}$% significance level, whether the mean length of the nails in the drum is greater than 2.5 cm.

b State where in the test the information that the nails have a normal distribution is used.

PS **5** Rahul has a six-sided die that he believes is biased so that the probability of rolling a 3 is smaller than $\frac{1}{6}$. He rolls the die 30 times and gets four 3s.

Does this provide sufficient evidence to support Rahul's belief?

PS **6** The 2011 census found that 68% of 16- to 19-year-olds in a particular town attended college. In 2015, a sample of 60 teenagers in this age range was surveyed and it was found that 46 of them attended college.

Is there evidence, at the 5% significance level, that the proportion of 16- to 19-year-olds attending college has increased?

PS **7** Ayesha is trying to find out whether it is true that students studying A Level Mathematics are more likely to be boys than girls. She sets up the following hypotheses:

$H_0 : p = 0.5$, $H_1 : p < 0.5$, where p is the proportion of girls studying A Level Mathematics.

She uses a sample of 30 A level students from her college, and decides to test her hypotheses at the 10% significance level.

Find the critical region for her test.

PS **8** A company is testing a new drug. They want to find out whether the drug cures a certain disease in more than 85% of cases.

a State suitable null and alternative hypotheses, defining any parameters.

The company decide to conduct their test at the 5% significance level, using a sample of 180 patients.

b Let X be the number of patients who are cured after using the drug. Find the critical region for the test.

PS **9** An established treatment for a particular disease is known to be effective in 82% of cases. A doctor devises a new treatment that she believes is even more effective. She uses the treatment on a random sample of 50 patients and finds that the new treatment is effective in 43 cases.

Does this data support the doctor's belief at the 2% significance level?

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PS **10** The drop-out rate of students enrolled at a certain university is reported to be 13.2%. The Dean of Students suspects that the drop-out rate for science students is greater than 13.2%, and he examines the records of a random sample of 95 of these students. The number of drop-outs was found to be 20. Test the Dean's suspicion at the 2% significance level.

PS **11** A sociologist believes that more boys than girls are born during war time. In a sample of 200 babies born in countries at war, 116 were boys.

Do these data support the sociologist's theory at the 5% significance level?

1.2 One-tailed and two-tailed hypothesis tests

WORKED EXAMPLE 1.2

A newspaper reported that 75% of students regularly cycle to college. A college dean believes that figure to be different at his college. He asks a sample of 160 students; 109 say they do cycle. Test the dean's belief at the 5% level of significance.

Answer

$H_0 : p = 0.75,\ H_1 : p \neq 0.75$ — State the hypotheses.

$X \sim B(160, 0.75) \approx N(120, 30)$

$P(X \leqslant 109) = P(X < 109.5) = \Phi\left(\dfrac{109.5 - 120}{\sqrt{30}}\right)$ — Remember to use a continuity correction.

$= \Phi(-1.917) = 1 - 0.972 = 0.0277$

$0.0277 > 0.025$ — For a two-tailed test compare the test result with half the significance level.

Accept H_0. There is insufficient evidence to show that the proportion cycling at the college is different. — Write your conclusion in context.

EXERCISE 1B

1 Write down the null and alternative hypotheses for the following tests, defining the meaning of any parameters.

a Sofia has a coin that she thinks is biased, and wants to use a hypothesis test to check this.

b Max knows that, last year, 26% of entries in AS Level Psychology were graded 'A' and wants to check whether this proportion has changed.

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2 In each question you are given null and alternative hypotheses (where p stands for the population proportion), the significance level and the observed data. Decide whether or not there is sufficient evidence to reject the null hypothesis.

a $H_0: p = 0.4,\ H_1: p \neq 0.4$, significance level 8%, observed 35 successes out of 100 trials

b $H_0: p = 0.8,\ H_1: p \neq 0.8$, significance level 5%, observed 17 successes out of 20 trials.

PS 3 A magazine article reported that 70% of computer owners use the internet regularly. Marie believed that the true figure was different and she consulted 12 of her friends who owned computers. Twelve said that they were regular users of the internet.

a Test Marie's belief at the 10% significance level.

b Comment on the reliability of the test in the light of Marie's sample.

PS 4 In the last general election, Party Z won 36% of the vote. An opinion poll surveys 100 people and finds that 45 support Party Z.

Does this provide sufficient evidence at the 10% significance level that the proportion of voters who support Party Z has changed?

PS 5 A teacher knows that in his old school, a third of all final-year students had a younger sibling at the school. He moves to a new school and wants to find out whether this proportion is different. He asks a sample of 60 final-year students, and finds that 27 of them have a younger sibling at the school.

Conduct a hypothesis test at the 5% significance level to decide whether there is evidence that the proportion of final-year students with a younger sibling at the new school is different to that of the old school.

PS 6 A large athletics club had the same running coach for several years. Records show that 28% of his athletes could run 100 metres in under 12 seconds. The club brings in a new coach and over the following year 26 out of a sample of 75 athletes recorded 100-metre times under 12 seconds.

Do these data support the hypothesis that the proportion of athletes who can run 100 metres in under 12 seconds has changed? Use the 5% significance level for your test.

PS 7 A jar contains a large number of coloured beads, some of which are red. A random sample of 80 of these beads is selected and 19 are found to be red. Test, at the 10% significance level, whether 30% of the beads in the jar are red.

PS 8 In order to test a coin for bias it is tossed 12 times. The result is 9 heads and 3 tails. Test, at the 10% significance level, whether the coin is biased.

PS 9 If births are equally likely on any day of the week then the proportion of babies born at the weekend should be $\frac{2}{7}$. Out of a random sample of 490 children it was found that 132 were born at the weekend. Does this provide evidence, at the 5% significance level, that the proportion of babies born at the weekend differs from $\frac{2}{7}$?

PS 10 A test is constructed to see if a coin is biased. It is tossed 10 times and if there are 10 heads, 9 heads, 1 head or 0 heads it is declared to be biased.

For each of the following, explain whether or not it could be the significance level for this test:

a 1% b 2% c 10% d 20%

1.3 Type I and Type II errors

WORKED EXAMPLE 1.3

A researcher claims that when people estimate values, 70% of the time the values estimated are a multiple of 10. Jinan believes the percentage should be higher. She conducts her own experiment with a sample of 19 people and tests the claim at the 5% significance level.

- **a** Show that the claim will be rejected if the estimates of 17 people are a multiple of 10, but not if the estimates of 16 people are a multiple of 10.
- **b** Calculate the probability of a Type I error.
- **c** If the proportion of people who estimate a multiple of 10 is 80%, find the probability that the test will result in a Type II error.

Answer

a $X \sim \mathrm{B}(19, 0.7)$ — $p = 0.7$.

$\mathrm{H}_0 : p = 0.7,\ \mathrm{H}_1 : p > 0.7$ — It is a one-tailed test because it is believed that the percentage should be higher.

$$\mathrm{P}(X \geqslant 17) = \binom{19}{17}0.7^{17}0.3^{2} + \binom{19}{18}0.7^{18}0.3^{1} + \binom{19}{19}0.7^{19} = 0.0462$$

Calculate the test statistics.

$$\mathrm{P}(X \geqslant 16) = \binom{19}{16}0.7^{16}0.3^{3} + 0.0462 = 0.133$$

Use the previous working if you can.

$0.0462 < 0.05$, whereas $0.133 > 0.05$

H_0 is rejected if $X \geqslant 17$, but not if $X \geqslant 16$.

A statement fully answers the question and shows you understand the calculations.

b $\mathrm{P}(\text{Type I error}) = \mathrm{P}(X \geqslant 17 \mid p = 0.7) = 0.0462$

For a Type I error, H_0 will be rejected when it should be accepted. We have found this probability in part **a**.

c $X \sim \mathrm{B}(19, 0.8)$ — Use $p = 0.8$.

$$\mathrm{P}(X < 17) = 1 - \mathrm{P}(X \geqslant 17) = 1 - \left(\binom{19}{17}0.8^{17}0.2^{2} + \binom{19}{18}0.8^{18}0.2^{1} + \binom{19}{19}0.8^{19}\right) = 1 - 0.237 = 0.763$$

For a Type II error, H_0 is accepted when it should be rejected. H_0 is rejected for $X \geqslant 17$ so we need to find $\mathrm{P}(X < 17)$.

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EXERCISE 1C

PS **1** A supplier of orchid seeds claims that their germination rate is 0.95. A purchaser of the seeds suspects that the germination rate is lower than this. In order to test this claim the purchaser plants 20 seeds in similar conditions and counts the number, X, which germinate. He rejects the claim if $X \leqslant 17$.

- **a** Formulate suitable null and alternative hypotheses to test the seed supplier's claim.
- **b** What is the probability of a Type I error using this test?
- **c** Calculate P(Type II error) if the probability that a seed germinates is in fact 0.80.

PS **2** A manufacturer claims that the probability that an electric fuse is faulty is no more than 0.03. A purchaser tests this claim by testing a box of 500 fuses. A significance test is carried out at the 5% level using X, the number of faulty fuses in a box of 500, as the test statistic.

- **a** For what values of X would you conclude that the probability that a fuse is faulty is greater that 0.03?
- **b** Estimate P(Type I error) for this test.
- **c** For this test estimate P(Type II error) if the probability that a fuse is faulty is, in fact, 0.06.

PS **3** A newspaper reported that 55% of households own more than one television set. Each of a random sample of 12 households in a town is contacted and the number of households owning more than one television set is denoted by N. A test of whether the proportion p of households in the town owning more than one television set is greater than 55% is carried out. It is decided to accept that p is greater than 55% if $N > 9$.

- **a** Calculate P(Type I error).
- **b** Calculate P(Type II error) when the actual value of p is 60%.

PS **4** It is suspected that the die used in a board game is biased away from a 6. In order to test this theory, the die is rolled 30 times and the number, X, of 6s is counted. If the number of 6s is less than 3 it is accepted that the die is biased away from a 6.

- **a** Set up suitable null and alternative hypotheses for testing the theory that the die is biased away from a 6.
- **b** Calculate the significance level of the test.
- **c** State the probability of a Type I error.
- **d** If, in fact, the probability of getting a 6 with the die is 0.1, calculate the probability of a Type II error.

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PS **5** A drug for treating phlebitis has proved effective in 75% of cases when it has been used. A new drug has been developed which, it is believed, will be more successful and it is used on a sample of 16 patients with phlebitis. A test is carried out to determine whether the new drug has a greater success rate than 75%. The test statistic is X, the number of patients cured by the new drug. It is decided to accept that the new drug is more effective if $X > 14$.

a Find α, the probability of making a Type I error.

b Find β, the probability of making a Type II error when the actual success rate is 80%.

PS **6** Of a certain make of electric toaster, 10% have to be returned for repair within three months of purchase. A modification to the toaster is made in the hope that it will be more reliable. Out of 24 modified toasters sold in a store, none was returned for repair within three months of purchase. The proportion of all the modified toasters that are returned for repair within three months of purchase is denoted by p.

a State, in terms of p, suitable hypotheses for a test.

b Test whether there is evidence, at a nominal 10% significance level, that the modified toaster is more reliable than the previous model in that it requires fewer repairs.

c What is the probability of making a Type I error in the test?

PS **7** It is known that many crimes are committed by people with backgrounds of drug abuse. A proportion of 60% has been suggested and, to investigate this, a researcher undertakes a study of 100 criminals and will carry out a test at a nominal 10% significance level. The null hypothesis is that the proportion of such criminals is 60% and the alternative hypothesis is that the proportion differs from 60%.

a Find the rejection region of the test.

b Find P(Type I error) for the test.

c Find P(Type II error) for the test when the actual proportion is 40%.

PS **8** An investigator suspects that operatives using a spring balance are reluctant to give 0 as the last value of a recorded weight, for example, 4.10 or 0.30. In order to test her theory she takes a random sample of 40 recorded weights and counts the number, X, which end in 0.

a State suitable hypotheses, involving a probability, for a hypothesis test which could indicate whether the operatives avoid ending a recorded weight with 0.

b Show that, for a test at the 10% significance level, the null hypothesis will be rejected if $X = 1$ but not if $X = 2$.

c State the rejection region for the test in terms of X.

d Calculate the value of P(Type I error).

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END-OF-CHAPTER REVIEW EXERCISE 1

PS **1** Angela is playing a board game with her friends, but thinks the die is biased and that a 6 is rolled too infrequently. In the subsequent 40 rolls of the die she got only three 6s.

Test Angela's belief at the 10% significance level.

PS **2** The proportion of 18- to 19-year-old students in favour of a new uniform is known to be 70%. Rhianna wants to find out whether the proportion of 16- to 17-year-old students is the same. She proposes to test the null hypothesis $H_0 : p = 0.7$ against two different alternative hypotheses, $H_1 : p < 0.7$ and $H_2 : p \neq 0.7$, using the 10% significance level.

Rhianna asks a sample of 25 16- to 17-year-old students. The data give her sufficient evidence to reject H_0 in favour of H_1, but not in favour of H_2.

How many students in Rhianna's sample were in favour of the new uniform?

PS **3** A student tests the hypothesis $H_0 : p = 0.4$ against $H_1 : p > 0.4$, where p is the proportion of brown cats of a particular breed.

In a sample of 80 cats of this breed, 40 were brown. This leads him to reject the null hypothesis.

What can you say about the significance level he used for his test?

PS **4** Sean has an eight-sided die and wants to check whether it is biased by looking at the probability, p, of rolling a 4. He sets up the following hypotheses:

$$H_0 : p = \frac{1}{8}, \ H_1 : p \neq \frac{1}{8}$$

To test them he decides to roll the die 80 times and reject the null hypothesis if the number of 4s is greater than 15 or fewer than 5.

a Let X be the number of 4s observed out of the 80 rolls.

State the name given to the region $5 \leqslant X \leqslant 15$.

b What is the probability that John incorrectly rejects the null hypothesis?

PS **5** A new cold relief drug is tested for effectiveness on 150 volunteers, and 124 of them found the drug beneficial. The manufacturers believe that more than 75% of people suffering from a cold will find the drug beneficial. Test the manufacturers' belief at the $2\frac{1}{2}$% significance level.

PS **6** A supermarket buys a large batch of plastic bags from a manufacturer to be used in the store. In previous batches 7% of the bags were defective. A quality control manager wishes to test whether the batch has a higher defective rate than 7%, in which case the batch will be returned to the manufacturer. He examines 125 randomly selected bags and finds that 14 are defective. Carry out the manager's test at the 3% significance level and state whether he should return the batch.

PS **7** A doctor knows that 20% of people suffer from side effects when treated with a certain drug. He wants to see if the proportion of people suffering from side effects is lower with a new drug. He looks at a random sample of 30 people treated with the new drug.

What is the largest number of people suffering from side effects that would still allow the doctor to conclude at 5% significance that the new drug has a lower proportion of side effects?

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PS **8** Aneka is investigating attitudes to sport among students at her school. She decides to carry out a survey using a sample of 70 students.

One of Aneka's questions is about participation in school sports teams. She wants to find out whether more than 40% of students play for a school team. She sets up the following hypotheses: $H_0: p = 0.4$, $H_1: p > 0.4$, where p is the proportion of students who play for a school sports team.

a Find the critical region for the hypothesis test at the 10% significance level, using a sample of 70 students.

b What is the probability of incorrectly rejecting the null hypothesis?

c In Aneka's sample, 32 students play for a school team. State the conclusion of the test.

9 The proportion of students getting an A in AS Mathematics is currently 33%. A publisher produces a new textbook that they hope will lead to improved performance. They trial their textbook with a sample of 120 students and want to test their hypothesis at the 5% significance level.

Find the critical region for this test.

10 A machine produces smartphone parts. Previous experience suggests that, on average, 7 in every 200 parts are faulty. After the machine was accidentally moved, a technician suspects that the proportion of faulty parts may have increased. She decides to test this hypothesis using a random sample of 85 parts.

a State suitable null and alternative hypotheses.

The technician decides that the critical region for the test should be $X \geqslant 5$. After checking her sample, she finds that 4 parts are faulty.

b State what conclusion she should draw and justify your answer.

c What is the probability of incorrectly rejecting the null hypothesis?

