Cambridge University Press
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Chapter 1
Quadratics

- Carry out the process of completing the square for a quadratic polynomial $ax^2 + bx + c$ and use a completed square form.
- Find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant.
- Solve quadratic equations, and quadratic inequalities, in one unknown.
- Solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic.
- Recognise and solve equations in x that are quadratic in some function of x.
- Understand the relationship between a graph of a quadratic function and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations.

1.1 Solving quadratic equations by factorisation

WORKED EXAMPLE 1.1

PS

Lim walked 12 km from A to B at a steady speed of x km/h.

His average speed for the return was 2 km/h slower.

a Write down, in terms of x, the total time taken for the complete journey.

b If the total time he took was 3.5 hours, write an equation in x and solve it to find his speed from A to B.

Answer

a $\frac{12}{x} + \frac{12}{x-2}$ hours — From using speed $= \frac{\text{distance}}{\text{time}}$.

b $\frac{12}{x} + \frac{12}{x-2} = 3.5$ — Multiply both sides by $x(x-2)$.

$12(x-2) + 12x = 3.5x(x-2)$ — Expand brackets and rearrange.

$12x - 24 + 12x = 3.5x^2 - 7x$ — Multiply both sides by 2 and rearrange.

$7x^2 - 62x + 48 = 0$ — Factorise.

$(7x-6)(x-8) = 0$ — Solve.

$x = \frac{6}{7}$ or $x = 8$

The value of $x = \frac{6}{7}$ is a solution to the equation, but it gives a negative speed for the return journey.

Lim's speed from A to B is 8 km/h.

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EXERCISE 1A

1 By factorising, solve the following equations:

a i $3x^2 + 2x = x^2 + 3x + 6$ ii $2x^2 + 3 = 17x - 7 - x^2$

b i $9x^2 = 24x - 16$ ii $18x^2 = 2x^2 - 40x - 25$

c i $(x - 3)(x + 2) = 14$ ii $(2x + 3)(x - 1) = 12$

d i $2x = 11 + \frac{6}{x}$ ii $3x + \frac{4}{x} = 7$

2 Solve the following equations. (In most cases, multiplication by an appropriate expression will turn the equation into a form you should recognise.)

a $x = 3 + \frac{10}{x}$ b $x + 5 = \frac{6}{x}$

c $2t + 5 = \frac{3}{t}$ d $x = \frac{12}{x + 1}$

e $x - \frac{2}{x + 2} = \frac{1}{3}$ f $\frac{20}{x + 2} - 1 = \frac{20}{x + 3}$

g $\frac{12}{x + 1} - \frac{10}{x - 3} = -3$ h $\frac{15}{2x + 1} + \frac{10}{x} = \frac{55}{2}$

3 Solve algebraically:

$(2x - 3)(x - 5) = (x - 3)^2$

4 Solve the equation $x^2 + 8k^2 = 6kx$, giving your answer in terms of k.

5 Find the exact solutions of the equation $x^2\sqrt{2} + 2x\sqrt{5} - 3\sqrt{2} = 0$.

6 Solve the equation $\frac{49}{(5x + 2)^2} - \frac{14}{5x + 2} + 1 = 0$

PS 7 The product of two positive, consecutive even integers is 168. Use this information to form a quadratic equation and solve it to find the two integers.

PS 8 Two men A and B working together can complete a task in 4 days. If B completes the task on his own, he takes 6 more days than if A did the task on his own.

Use the information to form an equation.

Solve the equation to find the time that A takes to complete the task on his own.

9 Solve by factorisation.

a $\frac{3x + 2}{2x - 1} = \frac{5x + 6}{x + 4}$ b $\frac{2}{3x + 1} + \frac{3}{1 - x} = \frac{1}{2}$

c $\frac{5}{x + 3} + \frac{7}{x - 1} = 8$ d $\frac{x^2 - 5x - 6}{x^2 - 1} = 0$

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PS **10** If 5 is a root of the equation $2x^2 - 3x + c = 0$, find the value of c and the second root of the equation.

1.2 Completing the square

WORKED EXAMPLE 1.2

a Write $x^2 + 6x - 5$ in completed square form.

b Complete the square $2x^2 + 12x - 5$.

c Express $1 - 4x - 2x^2$ in the form $p - q(x + r)^2$.

d **i** Express $x^2 - 7x + 15$ in the form $(x - a)^2 + b$ where a and b are constants.

ii Hence, state the maximum value of $\dfrac{1}{x^2 - 7x + 15}$.

Answer

a $(x^2 + 6x) - 5$ — Halve the coefficient of x and complete the square.

$\left\{(x + 3)^2 - 3^2\right\} - 5$ — Remove the brackets {}.

$(x + 3)^2 - 3^2 - 5$ — Simplify.

$(x + 3)^2 - 14$

b $2x^2 + 12x - 5$ — Take out the factor of 2 from the terms that involve x.

$2(x^2 + 6x) - 5$ — Complete the square.

$2\left\{(x + 3)^2 - 3^2\right\} - 5$ — Simplify.

$2(x + 3)^2 - 18 - 5$ — Simplify.

$2(x + 3)^2 - 23$

c $1 - 4x - 2x^2$ — Rearrange.

$-2x^2 - 4x + 1$ — Take out the factor of -2 from the terms that involve x.

$-2(x^2 + 2x) + 1$ — Complete the square.

$-2\left\{(x + 1)^2 - 1^2\right\} + 1$ — Simplify.

$-2(x+1)^2+2+1$ Simplify.

$-2(x+1)^2+3$ Rearrange.

$3-2(x+1)^2$

d **i** $x^2-7x+15$ Complete the square.

$$=\left(x-\frac{7}{2}\right)^2-\left(\frac{7}{2}\right)^2+15$$ Simplify.

$$=\left(x-\frac{7}{2}\right)^2+\frac{11}{4}$$

$a=\frac{7}{2}$, $b=\frac{11}{4}$

ii $$\frac{1}{\left(x-\frac{7}{2}\right)^2+\frac{11}{4}}$$ The **maximum** value of this fraction is when the denominator has the **minimum** value. This occurs when $x=\frac{7}{2}$.

$$=\frac{1}{\left(\frac{7}{2}-\frac{7}{2}\right)^2+\frac{11}{4}}$$ Evaluate the expression.

$$=\frac{4}{11}$$

EXERCISE 1B

1 Express the following in completed square form.

a x^2+2x+2 **b** x^2-8x-3 **c** x^2+3x-7

d $5-6x+x^2$ **e** $x^2+14x+49$ **f** $2x^2+12x-5$

g $3x^2-12x+3$ **h** $7-8x-4x^2$ **i** $2x^2+5x-3$

2 Use the completed square form to factorise the following expressions.

a $x^2-2x-35$ **b** $x^2-14x-176$ **c** $x^2+6x-432$

d $6x^2-5x-6$ **e** $14+45x-14x^2$ **f** $12x^2+x-6$

3 Solve the following quadratic equations. Leave surds in your answer.

a $(x-3)^2-3=0$ **b** $(x+2)^2-4=0$ **c** $2(x+3)^2=5$

d $(3x-7)^2=8$ **e** $(x+p)^2-q=0$ **f** $a(x+b)^2-c=0$

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PS **4** A recycling firm collects aluminium cans from a number of sites. It crushes them and then sells the aluminium back to a manufacturer.

The profit from processing t tonnes of cans each week is \$$p$, where

$$p = 100t - \frac{1}{2}t^2 - 200.$$

By completing the square, find the greatest profit the firm can make each week, and how many tonnes of cans it has to collect and crush each week to achieve this profit.

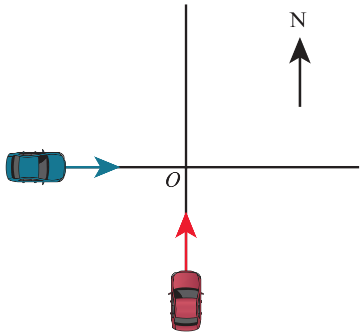
P **5** By writing the left-hand side in the form $a(x+p)^2 + q$, show that the equation $-2x^2 + 8x - 13 = 0$ has no real roots.

6 The quadratic function $y = a(x-b)^2 + c$ passes through the points $(-2, 0)$ and $(6, 0)$. Its maximum y value is 48. Find the values of a, b and c.

7 **a** Write $x^2 - 10x + 35$ in the form $(x-p)^2 + q$.

b Hence, or otherwise, find the maximum value of $\dfrac{1}{(x^2 - 10x + 35)^3}$.

PS **8** Two cars are travelling along two straight roads that are perpendicular to each other and meet at the point O, as shown in the diagram. The first car starts 50 km west of O and travels east at the constant speed of 20 km/h. At the same time, the second car starts 30 km south of O and travels north at the constant speed of 15 km/h.



a Show that at time t (hours) the distance d (km) between the two cars satisfies

$$d^2 = 625t^2 - 2900t + 3400.$$

b Hence find the closest distance between the two cars.

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1.3 The quadratic formula

WORKED EXAMPLE 1.3

Solve the equation $2x^2 - 2x - 1 = 0$. Write your answers as exact values (in surd form).

Answer

$2x^2 - 2x - 1 = 0$ Using $a = 2, b = -2$ and $c = -1$ in the quadratic formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 2 \times (-1)}}{2 \times 2}$$ Simplify.

$$x = \frac{2 + \sqrt{12}}{4} \text{ or } x = \frac{2 - \sqrt{12}}{4}$$

$$x = \frac{2 + 2\sqrt{3}}{4} \text{ or } x = \frac{2 - 2\sqrt{3}}{4}$$ Simplify the surd.

$$x = \frac{1 + \sqrt{3}}{2} \text{ or } x = \frac{1 - \sqrt{3}}{2}$$

EXERCISE 1C

1 Use the quadratic formula to find the exact solutions of the following equations:

a i $2x^2 + x = x^2 + 4x - 1$ ii $x^2 - 3x + 5 = 6 - 2x$

b i $3x^2 - 4x + 1 = 5x^2 + 2x$ ii $9x - 2 = 5x^2 + 1$

c i $(x + 1)(x + 3) = 5$ ii $(3x + 2)(x - 1) = 2$

d i $2x + \frac{1}{x} = 6$ ii $x = 4 + \frac{3}{x}$

2 Use the quadratic formula to solve the following equations. Leave irrational answers in surd form. If there is no solution, say so.

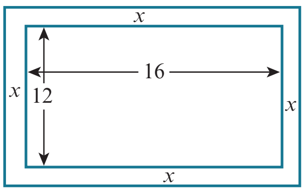
a $x^2 + 3x - 5 = 0$ b $x^2 - 4x - 7 = 0$ c $x^2 + 6x + 9 = 0$

d $x^2 + 5x + 2 = 0$ e $x^2 + x + 1 = 0$ f $3x^2 - 5x - 6 = 0$

g $2x^2 + 7x + 3 = 0$ h $8 - 3x - x^2 = 0$ i $5 + 4x - 6x^2 = 0$

PS 3 A rectangular garden measures 12 m by 16 m. A path is to be constructed around the perimeter of the garden (see diagram). The area of the garden plus path will be 285 m^2.

What will be the width of the path?



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PS **4** A square has sides of length $x + 2$ cm.

A right-angled isosceles triangle has its two equal sides of length $2x + 1$ cm.

The area of the square is equal to the area of the triangle.

By writing and solving a quadratic equation, find the perimeter of the square to 3 significant figures.

PS **5** The height of an object in metres above the ground is given by: $h = -16t^2 + 64t + 190,\ t \geqslant 0$ where t is the time in seconds.

Find the time it takes for the object to fall to the ground. Give your answer to 3 significant figures.

1.4 Solving simultaneous equations (one linear and one quadratic)

WORKED EXAMPLE 1.4

PS A 160 cm length of wire has to be bent to form three small square shapes together with one larger square shape.

The total area of the four square shapes is 508 cm^2.

The smaller squares have side length x cm and the large square has side length y cm. Use this information to form two equations.

Hence find the dimensions of each square.

Answer

The total perimeter of the 4 squares is equal to 160 cm.

So $12x + 4y = 160$ -----------(1)

The total area of the 4 squares is 508 cm^2.

So $3x^2 + y^2 = 508$ ------------(2)

$12x + 4y = 160$	Divide by 4.
$3x + y = 40$	Rearrange to make y the subject.
$y = 40 - 3x$	Substitute into (2).
$3x^2 + (40 - 3x)^2 = 508$	Expand brackets.
$3x^2 + 1600 - 120x - 120x + 9x^2 = 508$	Simplify.
$12x^2 - 240x + 1092 = 0$	Divide by 12.
$x^2 - 20x + 91 = 0$	Factorise.

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$$(x-7)(x-13)=0$$

$$x=7 \text{ or } x=13$$

Substituting $x=7$ into equation (1) gives $y=19$

Substituting $x=13$ into equation (1) gives $y=1$

The solutions are: $x=7, y=19$ and $x=13, y=1$

BUT

As the 3 squares are smaller than the single square, there is a unique solution.

The smaller squares are $7\,\text{cm} \times 7\,\text{cm}$ and the larger square is $19\,\text{cm} \times 19\,\text{cm}$.

EXERCISE 1D

1 Solve the following pairs of simultaneous equations.

a $y=x+1, x^2+y^2=25$

b $x+y=7, x^2+y^2=25$

c $y=x-3, y=x^2-3x-8$

d $y=2-x, x^2-y^2=8$

e $2x+y=5, x^2+y^2=25$

f $y=1-x, y^2-xy=0$

g $7y-x=49, x^2+y^2-2x-49=0$

h $y=3x-11, x^2+2xy+3=0$

2 The line $y=x-4$ intersects the curve $y=x^2+6x$ at two points.

Find:

a the coordinates of the intersection points

b the length of the line joining the intersection points as an exact value

c the equation of the perpendicular bisector of the line that joins these points.

3 Find the coordinates of the points of intersection of the given straight lines with the given curves.

a $y=2x+1, y=x^2-x+3$

b $y=3x+2, x^2+y^2=26$

c $y=2x-2, y=x^2-5$

d $x+2y=3, x^2+xy=2$

e $3y+4x=25, x^2+y^2=25$

f $y+2x=3, 2x^2-3xy=14$

g $y=2x-12, x^2+4xy-3y^2=-27$

h $2x-5y=6, 2xy-4x^2-3y=1$

4 The sum of two numbers is 8 and their product is 9.75.

a Show that this information can be written as a quadratic equation.

b What are the two numbers?

5 Find the point where the line $y = 3 - 4x$ meets the curve $y = 4(4x^2 + 5x + 3)$.

6 The straight line $y = x - 1$ meets the curve $y = x^2 - 5x - 8$ at the points A and B.

The curve $y = p + qx - 2x^2$ also passes through the points A and B. Find the values of p and q.

7 The line $y = 6x + 1$ meets the curve $y = x^2 + 2x + 3$ at two points. Show that the coordinates of one of the points are $(2 - \sqrt{2}, 13 - 6\sqrt{2})$, and find the coordinates of the other point.

8 Solve the equations $xy + x = 0$, $x^2 + y^2 = 4$.

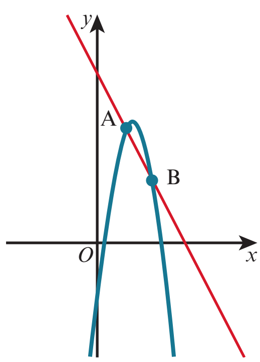
P 9 Curve P has equation $y = x^2 + kx - 3$.

Line L has equation $y = k - x$.

Prove that for all real values of k, the line L will intersect the curve P at exactly two points.

10 The figure shows part of the curve with equation $y = -x^2 + 10x + a$ (where a is a constant) and a straight line with equation $y = bx + 25$ (where b is a constant).

The x-coordinates of A and B are 4 and 8 respectively. Find the values of a and b.



1.5 Solving more complex quadratic equations

WORKED EXAMPLE 1.5

a Solve the equation $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$.

b Solve the equation $3\sqrt{x} + \dfrac{8}{\sqrt{x}} = 10$.

c Solve the equation $4y^{-2} + 1 - y^{-4} = 0$ giving your answers to 3 significant figures.

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Answer

a **Method 1:** Substitution method

$x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$ — Let $y = x^{\frac{1}{3}}$.

$y^2 - 2y - 15 = 0$

$(y - 5)(y + 3) = 0$

$y = 5$ or $y = -3$ — Substitute $x^{\frac{1}{3}}$ for y.

$x^{\frac{1}{3}} = 5$ or $x^{\frac{1}{3}} = -3$

$x = 5^3$ or $(-3)^3$

$x = 125$ or $x = -27$

Method 2: Factorise directly

$$x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$$

$$\left(x^{\frac{1}{3}} - 5\right)\left(x^{\frac{1}{3}} + 3\right) = 0$$

$x^{\frac{1}{3}} = 5$ or $x^{\frac{1}{3}} = -3$

$x = 125$ or $x = -27$

b $3\sqrt{x} + \dfrac{8}{\sqrt{x}} = 10$ — Multiply each term by $\sqrt{x}$.

$3x + 8 = 10\sqrt{x}$ — Rearrange.

$3x - 10\sqrt{x} + 8 = 0$ — Let $y = \sqrt{x}$.

$3y^2 - 10y + 8 = 0$ — Factorise.

$(3y - 4)(y - 2) = 0$

$y = \dfrac{4}{3}$ or $y = 2$ — Substitute $\sqrt{x}$ for y.

$\sqrt{x} = \dfrac{4}{3}$ or $\sqrt{x} = 2$

$\therefore x = \dfrac{16}{9}$ or 4

TIP

Remember that $x^{\frac{2}{3}} = \left(x^{\frac{1}{3}}\right)^2$