1
Introduction

1.1 Orthogonal polynomials on the real line

Orthonormal polynomials $(p_n)_{n\in\mathbb{N}}$ on the real line are defined by the orthogonality conditions

$$\int_{\mathbb{R}} p_n(x)p_m(x)\,d\mu(x) = \delta_{m,n}, \tag{1.1}$$

where μ is a positive measure on the real line for which all the moments exist and $p_n(x) = \gamma_n x^n + \cdots$, with positive leading coefficient $\gamma_n > 0$. A family of orthonormal polynomials always satisfies a three-term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n \geq 0, \tag{1.2}$$

with $p_{-1} = 0$ and

$$p_0 = \gamma_0 = (\mu(\mathbb{R}))^{-1/2}.$$

Comparing the leading coefficients in the recurrence relation gives

$$a_{n+1} = \frac{\gamma_n}{\gamma_{n+1}} > 0, \tag{1.3}$$

and computing the Fourier coefficients of $xp_n(x)$ in (1.2) gives

$$a_n = \int_{\mathbb{R}} xp_n(x)p_{n-1}(x)\,d\mu(x), \tag{1.4}$$

$$b_n = \int_{\mathbb{R}} xp_n^2(x)\,d\mu(x). \tag{1.5}$$

For the monic orthogonal polynomials $P_n = p_n/\gamma_n$ the recurrence relation is

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n^2 P_{n-1}(x), \tag{1.6}$$

where the b_n are as in (1.5) and the a_n^2 are the squares of (1.4).

The converse statement is also true and is known as *the spectral theorem for orthogonal polynomials*[1]: if a family of polynomials satisfies a three-term recurrence relation of the form (1.2), with $a_n > 0$ and $b_n \in \mathbb{R}$ and with initial conditions $p_0 = 1$ and $p_{-1} = 0$, then there exists a probability measure μ on the real line such that these polynomials are orthonormal polynomials satisfying (1.1). This gives rise to two important problems:

Problem 1. Suppose the measure μ is known. What can be said about the recurrence coefficients $(a_n)_{n=1,2,3,...}$ and $(b_n)_{n=0,1,2,...}$? This is known as the *direct problem for orthogonal polynomials*.

Problem 2. Suppose the recurrence coefficients $(a_{n+1}, b_n)_{n=0,1,2,...}$ are known. What can be said about the orthogonality measure μ? This is known as the *inverse problem for orthogonal polynomials*.

A partial solution of problem 1 is that one can express the recurrence coefficients a_n^2 and b_n in terms of the moments of the measure μ. Let

$$m_n = \int_{\mathbb{R}} x^n \, d\mu(x), \qquad n \geq 0,$$

and define the Hankel determinants

$$\Delta_{n+1} = \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & m_3 & \cdots & m_n & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+1} & m_{n+2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n-1} & m_{2n} \end{pmatrix}, \tag{1.7}$$

then the monic orthogonal polynomial P_n is given by

$$P_n(x) = \frac{1}{\Delta_n} \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & m_3 & \cdots & m_n & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+1} & m_{n+2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-2} & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} & x^n \end{pmatrix}. \tag{1.8}$$

From this one easily computes

$$\frac{1}{\gamma_n^2} = \int_{\mathbb{R}} P_n^2(x) \, d\mu(x) = \frac{\Delta_{n+1}}{\Delta_n},$$

[1] Also known as Favard's theorem, but the result is much older hence the attribution to Favard is not so appropriate.

so that from (1.3) one finds

$$a_n^2 = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}. \tag{1.9}$$

If we write $P_n(x) = x^n + \delta_n x^{n-1} + \cdots$ and compare the coefficients of x^n in (1.6), then one finds

$$b_n = \delta_n - \delta_{n+1}. \tag{1.10}$$

The coefficient δ_n can be obtained from (1.8) and is $\delta_n = -\Delta_n^*/\Delta_n$, where Δ_n^* is obtained from Δ_n by replacing the last column $(m_{n-1}, m_n, \ldots, m_{2n-2})^T$ by $(m_n, m_{n+1}, \ldots, m_{2n-1})^T$. One then has from (1.10)

$$b_n = \frac{\Delta_{n+1}^*}{\Delta_{n+1}} - \frac{\Delta_n^*}{\Delta_n}. \tag{1.11}$$

The formulas (1.9) and (1.11) however do not really show how properties of the measure μ can be transferred to properties of the recurrence coefficients. One needs more tools to solve this direct problem for orthogonal polynomials.

The recurrence coefficients are usually collected in a tridiagonal matrix of the form

$$J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \\ a_1 & b_1 & a_2 & 0 & 0 & \\ 0 & a_2 & b_2 & a_3 & 0 & \\ 0 & 0 & a_3 & b_3 & \ddots & \\ 0 & 0 & 0 & \ddots & \ddots & \end{pmatrix}, \tag{1.12}$$

which acts as an operator on (a subset of) $\ell^2(\mathbb{N})$ and which is known as the *Jacobi matrix* or Jacobi operator. If J is selfadjoint, then the spectral measure for J is precisely the orthogonality measure μ. Hence problem 1 corresponds to the inverse problem for the Jacobi matrix J and problem 2 corresponds to the direct problem for J.

In the present notes we will study problem 1 for a few special cases. In Chapter 2 we study measures on the real line with an exponential weight function of the form $d\mu(x) = |x|^\rho \exp(-|x|^m)\, dx$, which are known as Freud weights, named after Géza Freud who studied them in the 1970s. It will be shown that the recurrence coefficients $(a_n)_{n\geq 1}$ satisfy a nonlinear recurrence relation which corresponds to the discrete Painlevé I equation and its hierarchy. In Chapter 3 we will study a family of orthogonal polynomials on the unit circle. We will first give some background on orthogonal polynomials on the unit circle and the corresponding recurrence relations. We will study the weight function $w(\theta) = \exp(t\cos\theta)$, and it will be shown that the recurrence coefficients satisfy a nonlinear recurrence relation which corresponds to discrete Painlevé

II. These orthogonal polynomials play an important role in the theory of random unitary matrices and combinatorial problems for random permutations. We will also study certain discrete orthogonal polynomials related to Charlier polynomials. The recurrence coefficients $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ are shown to satisfy a system of nonlinear recurrence relations which are again related to the discrete Painlevé II equation. In Chapter 4 we give some details about ladder operators for orthogonal polynomials. These are (differential or difference) operators that map an orthogonal polynomial of degree n to one of degree $n-1$ (lowering operator) or degree $n+1$ (raising operator). The compatibility of these ladder operators with the three term recurrence relation gives nonlinear recurrence relations for the recurrence coefficients (a_n, b_n), which can often be identified as discrete Painlevé equations. In Chapter 5 we give some more examples of semi-classical orthogonal polynomials that give rise to discrete and continuous Painlevé equations. In Chapters 6 and 7 we will investigate the six (differential) Painlevé equations but restrict our attention to those aspects that involve orthogonal polynomials. In Chapter 6 we investigate solutions of the Painlevé equations which are in terms of orthogonal polynomials. These are rational solutions and solutions in terms of classical (linear) special functions. Finally, in Chapter 7 we show how Painlevé transcendents appear in the asymptotic analysis of orthogonal polynomials near critical points, i.e., points where the density of the zeros vanishes or becomes unbounded. Usually this corresponds to a phase transition: the zero density is supported on a number of intervals and when these intervals touch or when a new interval appears one often has singular behavior of the zero density.

1.1.1 Pearson equation and semi-classical orthogonal polynomials

Classical orthogonal polynomials are orthogonal with a weight function w on the real line which satisfies a first order differential equation

$$(\sigma w)' = \tau w, \tag{1.13}$$

where σ is a polynomial of degree ≤ 2 and τ a polynomial of degree 1. This equation is known as the *Pearson equation*, named after the statistician Karl Pearson who introduced it in 1895. We are interested in positive solutions w such that σw vanishes at points $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. Up to an affine transformation, the classical orthogonal polynomials are

- The *Hermite polynomials*, with $w(x) = e^{-x^2}$ on $(-\infty, \infty)$ and $\sigma = 1$;
- The *Laguerre polynomials*, with $w(x) = x^\alpha e^{-x}$ on $[0, \infty)$ and $\sigma(x) = x$ $(\alpha > -1)$;

- The *Jacobi polynomials*, with $w(x) = (1-x)^\alpha(1+x)^\beta$ on $[-1,1]$ and $\sigma(x) = x^2-1$ $(\alpha,\beta > -1)$.

The case $\sigma(x) = x^2$ is related to Bessel polynomials but does not give orthogonal polynomials with a positive measure on the real line. The case $\sigma(x) = x^2+1$ gives Romanovski polynomials, but then we can only have a finite number of orthogonal polynomials with a positive measure on the real line, see [101, Thm. 4.1].

Semi-classical orthogonal polynomials have a weight function w that satisfies a Pearson equation (1.13) where σ and τ are polynomials with $\deg \sigma > 2$ or $\deg \tau \neq 1$. We need positive solutions w such that σw vanishes at $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. An important property of classical and semi-classical orthogonal polynomials is their *structure relation*:

Property 1.1 *If the weight w satisfies the Pearson equation* (1.13) *and σw vanishes at $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, then*

$$\sigma(x)p_n'(x) = \sum_{k=n-t}^{n+s-1} A_{n,k}p_k(x), \tag{1.14}$$

where $s = \deg \sigma$ and $t = \max\{\deg \tau, \deg \sigma - 1\}$.

Proof The polynomial $\sigma p_n'$ has degree $n+s-1$, so we can expand it in terms of the orthonormal polynomials p_k with $0 \le k \le n+s-1$:

$$\sigma(x)p_n'(x) = \sum_{k=0}^{n+s-1} A_{n,k}p_k(x).$$

The coefficients $A_{n,k}$ are Fourier coefficients and can be expressed as

$$A_{n,k} = \int_a^b \sigma(x)p_n'(x)p_k(x)w(x)\,dx.$$

Integration by parts, and the boundary conditions $\sigma(a)w(a) = 0 = \sigma(b)w(b)$, gives

$$\begin{aligned}
A_{n,k} &= -\int_a^b p_n(x)[\sigma(x)w(x)p_k(x)]'\,dx \\
&= -\int_a^b p_n(x)p_k(x)[\sigma(x)w(x)]'\,dx - \int_a^b p_n(x)p_k'(x)\sigma(x)w(x)\,dx \\
&= -\int_a^b p_n(x)p_k(x)\tau(x)w(x)\,dx - \int_a^b p_n(x)p_k'(x)\sigma(x)w(x)\,dx,
\end{aligned}$$

where we used the Pearson equation (1.13) in the last line. By orthogonality the first term vanishes whenever $k + \deg \tau < n$ and the second term vanishes

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whenever $k + s - 1 < n$, hence both terms vanish whenever $k < n - t$ with $t = \max\{\deg \tau, \deg \sigma - 1\}$ and only the Fourier coefficients $A_{n,k}$ with $n - t \leq k \leq n + s - 1$ are left. □

Every family of orthogonal polynomials on the real line satisfies a three term recurrence relation (1.2) and semi-classical orthogonal polynomials in addition also satisfy a structure relation (1.14). Both relations should be *compatible*. If we express the compatibility relations in terms of the recurrence coefficients $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 0}$ and the coefficients $(A_{n,k})_{n\geq 1}$ in the structure relation, then we get (nonlinear) recurrence relations for these coefficients. Solving them gives the recurrence coefficients $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$.

To illustrate this we use the Hermite polynomials, for which $w(x) = e^{-x^2}$ on $(-\infty, +\infty)$ and $\sigma = 1$. The structure relation for the orthonormal Hermite polynomials is

$$p_n'(x) = A_n p_{n-1}(x).$$

Taking derivatives in the three term recurrence relation (1.2) gives

$$p_n(x) + xp_n'(x) = a_{n+1}p_{n+1}'(x) + b_n p_n'(x) + a_n p_{n-1}'(x).$$

Use the structure relation to replace all the derivatives, then

$$p_n(x) + A_n x p_{n-1}(x) = a_{n+1}A_{n+1}p_n(x) + b_n A_n p_{n-1}(x) + a_n A_{n-1}p_{n-2}(x).$$

Now replace $xp_{n-1}(x)$ by using the three term recurrence relation, to find

$$\begin{aligned} p_n(x) + A_n[a_n p_n(x) + b_{n-1}p_{n-1}(x) + a_{n-1}p_{n-2}(x)] \\ = a_{n+1}A_{n+1}p_n(x) + b_n A_n p_{n-1}(x) + a_n A_{n-1}p_{n-2}(x). \end{aligned}$$

Since the orthogonal polynomials $\{p_n, p_{n-1}, p_{n-2}\}$ are linearly independent, this expression can only be true if the coefficients in front of p_n, p_{n-1} and p_{n-2} vanish. This gives three equations

$$p_n \Rightarrow 1 + A_n a_n = a_{n+1}A_{n+1}, \tag{1.15}$$

$$p_{n-1} \Rightarrow A_n b_{n-1} = b_n A_n, \tag{1.16}$$

$$p_{n-2} \Rightarrow A_n a_{n-1} = a_n A_{n-1}. \tag{1.17}$$

From (1.16) we find that $b_n = b_{n-1}$ so that b_n is a constant sequence: $b_n = b_0$. From (1.15) we find $a_{n+1}A_{n+1} - a_n A_n = 1$ so that $a_n A_n = n + a_0 A_0$, but we defined $p_{-1} = 0$ so that (1.4) gives $a_0 = 0$. Hence $a_n A_n = n$. Finally (1.17) gives $A_n/a_n = A_{n-1}/a_{n-1}$ so that $A_n/a_n = c$ is constant. Combining this with the previous relation gives $a_n^2 = n/c$ for $n \geq 1$, so that $1/c = a_1^2$. So for Hermite polynomials we were able to solve the nonlinear equations to find

$$b_n = b_0, \qquad a_n^2 = a_1^2 n.$$

We only need to figure out what the initial values b_0 and a_1^2 are, to get all the recurrence coefficients. For b_0 we use (1.5) with $n = 0$ to find

$$b_0 = p_0^2 \int_{-\infty}^{\infty} x e^{-x^2}\, dx = 0,$$

so that $b_n = 0$ for all $n \geq 0$. In fact, this was already clear from the beginning since $w(x) = e^{-x^2}$ is an even weight function. For a_1^2 we use the fact that $p_1(x) = (x - b_0)p_0/a_1$ has norm 1. Recall that

$$p_0^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)^{-1} = 1/\sqrt{\pi}$$

hence

$$1 = \int_{-\infty}^{\infty} p_1^2(x) e^{-x^2}\, dx = \frac{1}{a_1^2 \sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \frac{1}{2a_1^2},$$

so that $a_1^2 = 1/2$ and $a_n^2 = n/2$ for $n \geq 1$. Hence the three term recurrence relation for orthonormal Hermite polynomials is

$$\sqrt{2} x p_n(x) = \sqrt{n+1}\, p_{n+1}(x) + \sqrt{n}\, p_{n-1}(x)$$

and the structure relation is (recall that $A_n = a_n/a_1^2$)

$$p_n'(x) = \sqrt{2n}\, p_{n-1}(x).$$

Note that the usual Hermite polynomials $(H_n)_{n\in\mathbb{N}}$ are not orthonormal, but have norm

$$\int_{-\infty}^{\infty} H_n^2(x) e^{-x^2}\, dx = \sqrt{\pi} 2^n n!,$$

hence $H_n(x) = \sqrt{\sqrt{\pi} 2^n n!}\, p_n(x)$. The corresponding recurrence relation and structure relation then become

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x), \qquad H_n'(x) = 2nH_{n-1}.$$

Exercise 1: Use the compatibility relations between (1.2) and (1.14) to find the recurrence coefficients for the orthonormal Laguerre polynomials with weight function $w(x) = x^\alpha e^{-x}$ on $[0, \infty)$ for $\alpha > -1$.

1.2 Painlevé equations

1.2.1 The six Painlevé differential equations

Linear differential equations are reasonably easy to investigate. Nonlinear differential equations are a lot harder and several problems, which do not exist for linear equations, appear. One such problem is that the singularities of the solution may depend on the initial conditions. Such singularities are called *movable singularities*. For instance $y' = -y^2$ has the general solution $y(x) = 1/(x-c)$ where c is constant, hence the singularity at $x = c$ depends on the constant of integration, or on the initial value: $c = -1/y(0)$. The movable singularity is a pole in this case. This is not so bad, because it is an isolated singularity. The equation $y' = 1/(2y)$ has the general solution $y(x) = \sqrt{x-c}$ with c a constant. Now the singularities are on a half line in the complex plane starting at c. This is a branch cut and c is a branch point and the location of this branch point depends on the initial condition $c = -y(0)^2$. The movable singularities are not poles but more complicated and depend on the choice of the branch cut. This situation is not desirable and may lead to serious complications when we are comparing solutions of differential equations. Hence, at the end of the 19th century people (Poincaré, Fuchs, Picard, Painlevé) became interested in finding those nonlinear differential equations for which **the general solution is free from movable branch points**. This is called the *Painlevé property*. The locations of possible branch points and critical essential singularities of solutions may not depend on the initial values. For first order differential equations the Painlevé property only gives linear differential equations, the Weierstrass elliptic function $\wp$ satisfying $(y')^2 = 4y^3 - g_2 y - g_3$ or the Riccati differential equation $y' = q_0(x) + q_1(x)y + q_2(x)y^2$. Picard raised the problem of finding the nonlinear differential equations of the form $y'' = R(y', y, x)$, where R is a rational function, with the Painlevé property. At the beginning of the 20th century Paul Painlevé found that, up to certain simple transformations, these differential equations can be put into one of 50 canonical forms. Out of these 50, there are 44 that can be reduced to linear equations, the Weierstrass elliptic equation, the Riccati equation, or one of six equations of the list. These six equations are now known as the Painlevé equations and their solutions are called *Painlevé transcendents*. It turns out that for these second order equations the only movable singularities are poles (no essential singularities). These Painlevé equations are important nonlinear special functions that nowadays appear in integrable systems, statistical mechanics, random matrix theory and orthogonal polynomials. The six equations are

$$\mathrm{P_I} \quad y'' = 6y^2 + x, \tag{1.18}$$

$$\mathrm{P_{II}} \quad y'' = 2y^3 + xy + \alpha, \tag{1.19}$$

$$\mathrm{P_{III}} \quad y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{1.20}$$

$$\mathrm{P_{IV}} \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \tag{1.21}$$

$$\mathrm{P_V} \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}, \tag{1.22}$$

$$\mathrm{P_{VI}} \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right), \tag{1.23}$$

where $\alpha, \beta, \gamma, \delta$ are constants. A good survey can be found in [39].

1.2.2 Discrete Painlevé equations

Discrete Painlevé equations appeared more recently. They are nonlinear "integrable" discrete equations (recurrence relations) for which the continuous limit is one of the Painlevé differential equations. Usually only second order equations are considered. The term *integrable* remains ambiguous: what do we mean by that? Without giving a precise meaning it basically means that anything simpler becomes linear, anything more complicated becomes hopelessly complicated. For a good survey we refer to [80].

There is a discrete version of the Painlevé property which one can use as a detector for integrability. This notion is *singularity confinement*. Suppose that we are dealing with a recurrence relation $x_n = f(x_{n-2}, x_{n-1}, n)$, with f a rational function. Let n_0 be an index such that $(x_{n_0-2}, x_{n_0-1}, n_0)$ gives a singularity for f, so that x_{n_0} is not defined. Then singularity confinement means that there is an integer p such that the singularity is confined to the elements $x_{n_0}, x_{n_0+1}, \ldots, x_{n_0+p}$ but x_{n_0+p+1} is again defined and it depends on what happened before the singularities, i.e., on x_{n_0-1}. So the singularity is restricted to a finite section of the sequence $(x_n)_{n\in\mathbb{N}}$, which is the discrete version of an isolated singularity (a pole) for complex functions. To check for singularity confinement, one usually starts from $\tilde{x}_{n_0-1} = x_{n_0-1} + \epsilon$ and one computes

$$\begin{aligned} \tilde{x}_{n_0} &= O(\frac{1}{\epsilon}) + \cdots \\ &\vdots \\ \tilde{x}_{n_0+p} &= O(\frac{1}{\epsilon}) + \cdots \\ \tilde{x}_{n_0+p+1} &= x_{n_0+p+1} + O(\epsilon) \end{aligned}$$

with a careful analysis of the error terms. Then as $\epsilon \to 0$ we can find the value of p and we can see how x_{n_0+p+1} depends on the past (before the singularity). The property of singularity confinement however *does not* characterize discrete Painlevé equations: there are examples of discrete equations with singularity confinement, which we should not call a discrete Painlevé equation. For this reason, singularity confined is only used as a discrete integrability detector.

Making a canonical list of discrete Painlevé equations is more complicated than for differential equations since we cannot use transformations of the variable n to construct an equivalence class of equations of the same type. We still can use transformations of the solution, of course. A list of standard discrete Painlevé equations grew historically as the equations appeared. A partial list is

$$\text{d-P}_{\text{I}} \quad x_{n+1} + x_n + x_{n-1} = \frac{z_n + a(-1)^n}{x_n} + b, \tag{1.24}$$

$$\text{d-P}_{\text{II}} \quad x_{n+1} + x_{n-1} = \frac{x_n z_n + a}{1 - x_n^2}, \tag{1.25}$$

$$\text{d-P}_{\text{IV}} \quad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n + z_n)^2 - c^2}, \tag{1.26}$$

$$\begin{aligned} \text{d-P}_{\text{V}} \quad & \frac{(x_{n+1} + x_n - z_{n+1} - z_n)(x_n + x_{n-1} - z_n - z_{n-1})}{(x_{n+1} + x_n)(x_n + x_{n-1})} \\ &= \frac{[(x_n - z_n)^2 - a^2][(x_n - z_n)^2 - b^2]}{(x_n - c^2)(x_n - d^2)}, \end{aligned} \tag{1.27}$$

where $z_n = \alpha n + \beta$ and a, b, c, d are constants. Observe that there is no d-P_{III} or d-P_{VI}. That is because in the above equations the x_n and x_{n+1} (and x_n and x_{n-1}) appear in an additive way. There are other discrete Painlevé equations where