Probability on Graphs

Second Edition

This introduction to some of the principal models in the theory of disordered systems leads the reader through the basics to the very edge of contemporary research, with minimal technical fuss. Topics covered include random walks, percolation, self-avoiding walks, interacting particle systems, uniform spanning trees, and random graphs, as well as the Ising, Potts, and random-cluster models for ferromagnetism, and the Lorentz model for motion in a random medium. Schramm-Löwner evolutions (SLE) arise in various contexts. This new edition features topics in which there has been major recent progress, including the exact value of the connective constant of the hexagonal lattice and the critical point of the random-cluster model on the square lattice.

The choice of topics is strongly motivated by modern applications, and focuses on areas that merit further research. Special features include a simple account of Smirnov's proof of Cardy's formula for critical percolation, and an account of the theory of influence and sharp-thresholds. Accessible to a wide audience of mathematicians and physicists, this book can be used as a graduate course text. Each chapter ends with a range of exercises.

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Probability on Graphs

Random Processes on Graphs and Lattices

Second Edition

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Preface

Within the menagerie of objects studied in contemporary probability theory, a number of related animals have attracted great interest amongst probabilists and physicists in recent years. The inspiration for many of these objects comes from physics, but the mathematical subject has taken on a life of its own and many beautiful constructions have emerged. The overall target of these notes is to identify some of these topics, and to develop their basic theory at a level suitable for mathematics graduates.

If the two principal characters in these notes are random walk and percolation, they are only part of the rich theory of uniform spanning trees, self-avoiding walks, random networks, models for ferromagnetism and the spread of disease, and motion in random environments. This is an area that has attracted many fine scientists, by virtue, perhaps, of its special mixture of modelling and problem-solving. There remain many open problems. It is the experience of the author that these may be explained successfully to a graduate audience open to inspiration and provocation.

The material described here may be used for personal study and also as the bases of lecture courses of between 16 and 48 hours duration. Little is assumed about the mathematical background of the audience beyond some basic probability theory, but students should be willing to get their hands dirty if they are to profit. Care should be taken in the setting of examinations, since problems can be unexpectedly difficult. Successful examinations may be designed, and some help is offered through the inclusion of exercises at the ends of chapters. As an alternative to a conventional examination, students could be asked to deliver presentations on aspects and extensions of the topics studied.

Chapter 1 is devoted to the relationship between random walks (on graphs) and electrical networks. This leads to the Thomson and Rayleigh principles, and thence to a proof of Pólya's theorem. In Chapter 2, we describe Wilson's algorithm for constructing a uniform spanning tree (UST), and we discuss boundary conditions and weak limits for UST on a lattice. This chapter includes a brief introduction to Schramm–Löwner evolutions (SLEs).

Percolation theory appears first in Chapter 3, together with a short introduction to self-avoiding walks. Correlation inequalities and other general techniques are described in Chapter 4. A special feature of this part of the book is a fairly full treatment of influence and sharp-threshold theorems for product measures, and more generally for monotone measures.

We return to the basic theory of percolation in Chapter 5, including a full account of Smirnov's proof of Cardy's formula. This is followed in Chapter 6 by a study of the contact model on lattices and trees.

Chapter 7 begins with a proof of the equivalence of Gibbs states and Markov fields, and continues with an introduction to the Ising and Potts models. Chapter 8 contains an account of the random-cluster model. The quantum Ising model features in the next chapter, particularly through its relationship to a continuum random-cluster model and the consequent analysis using stochastic geometry.

Interacting particle systems form the basis of Chapter 10. This is a large field in its own right, and little is done here beyond introductions to the contact, voter, and exclusion models, and to the stochastic Ising model. Chapter 11 is devoted to random graphs of Erdős–Rényi type. There are accounts of the giant cluster, and of the chromatic number via an application of Hoeffding's inequality for the tail of a martingale.

The final Chapter 12 contains one of the most notorious open problems in stochastic geometry, namely the Lorentz model (or Ehrenfest wind–tree model) on the square lattice.

This text is based in part on courses given by the author within Part 3 of the Mathematical Tripos at Cambridge University over a period of several years. They have been prepared in the present form as background material for lecture courses presented to outstanding audiences of students and professors at the 2008 PIMS–UBC Summer School in Probability and during the programme on Statistical Mechanics at the Institut Henri Poincaré, Paris, during the last quarter of 2008. The book was written in part during a visit to the Mathematics Department at UCLA (with partial support from NSF grant DMS-0301795), to which the author expresses his gratitude for the warm welcome received there, and in part during programmes at the Isaac Newton Institute and the Institut Henri Poincaré–Centre Emile Borel.

Throughout this work, pointers are included to more extensive accounts of the topics covered. The selection of references is intended to be useful rather than comprehensive.

The author thanks four artists for permission to include their work: Tom Kennedy (Figure 2.1), Oded Schramm (Figures 2.2–2.4), Raphaël Cerf (Figure 5.2), and Julien Dubédat (Figure 5.17). The section on influence has ben-

efited from conversations with Rob van den Berg and Tom Liggett. Stanislav Smirnov and Wendelin Werner have consented to the inclusion of some of their neat arguments, hitherto unpublished. Several readers have proposed suggestions and corrections. Thank you, everyone!

G. R. G.
Cambridge
April 2010

Preface to the Second Edition

The major additions in this new edition include: a proof of the connective constant of the hexagonal lattice (Theorem 3.14), an improved influence theorem for general product spaces (Theorem 4.38), a streamlined proof of exponential decay for subcritical percolation (Theorem 5.1), and a proof of the critical point of the random-cluster model on the square lattice (Theorem 8.25).

The author is grateful to students and colleagues for their suggestions for improvements. Special thanks are due to Naser Talebizadeh Sardari, Claude Bélisle, Svante Janson, and Russell Lyons. Some of the writing was done during a visit to the Statistics Department of the University of California at Berkeley, with partial support from UC Berkeley and from the Engineering and Physical Science Research Council under grant EP/I03372X/1.

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