CAMBRIDGE
Cambridge University Press
978-1-108-43817-9 — Probability on Graphs
2nd Edition
Excerpt
More Information

1

Random Walks on Graphs

The theory of electrical networks is a fundamental tool for studying the recurrence of reversible Markov chains. The Kirchhoff laws and Thomson principle permit a neat proof of Pólya's theorem for random walk on a d-dimensional grid.

1.1 Random Walks and Reversible Markov Chains

A basic knowledge of probability theory is assumed in this volume. Readers keen to acquire this are referred to [150] for an elementary introduction, and to [148] for a somewhat more advanced account. We shall generally use the letter $\mathbb{P}$ to denote a generic probability measure, with more specific notation when helpful. The expectation of a random variable f will be written as either $\mathbb{P}(f)$ or $\mathbb{E}(f)$.

Only a little knowledge is assumed about graphs, and many readers will have sufficient acquaintance already. Others are advised to consult Section 1.6. Of the many books on graph theory, we mention [50].

Let $G = (V, E)$ be a finite or countably infinite graph, which we generally assume, for simplicity, to have neither loops nor multiple edges. If G is infinite, we shall usually assume in addition that every vertex-degree is finite. A particle moves around the vertex-set V. Having arrived at the vertex S_n at time n, its next position S_{n+1} is chosen uniformly at random from the set of neighbours of S_n. The trajectory of the particle is called a *symmetric random walk* (SRW) on G.

Two of the basic questions concerning symmetric random walk are:

1. Under what conditions is the walk *recurrent*, in that it returns (almost surely) to its starting point?
2. How does the distance between S_0 and S_n behave as $n \to \infty$?

The above SRW is symmetric in that the jumps are chosen *uniformly* from the set of available neighbours. In a more general process, we take a function $w : E \to (0, \infty)$, and we jump along the edge e with probability proportional to w_e.

Any reversible Markov chain[1] on the set V gives rise to such a walk as follows. Let $Z = (Z_n : n \geq 0)$ be a Markov chain on V with transition matrix P, and assume that Z is reversible with respect to some positive function $\pi : V \to (0, \infty)$, which is to say that

$$\pi_u p_{u,v} = \pi_v p_{v,u}, \qquad u, v \in V. \tag{1.1}$$

With each distinct pair $u, v \in V$, we associate the weight

$$w_{u,v} = \pi_u p_{u,v}, \tag{1.2}$$

noting by (1.1) that $w_{u,v} = w_{v,u}$. Then

$$p_{u,v} = \frac{w_{u,v}}{W_u}, \qquad u, v \in V, \tag{1.3}$$

where

$$W_u = \sum_{v \in V} w_{u,v}, \qquad u \in V.$$

That is, given that $Z_n = u$, the chain jumps to a new vertex v with probability proportional to $w_{u,v}$. This may be set in the context of a random walk on the graph with vertex-set V and edge-set E containing all $e = \langle u, v \rangle$ such that $p_{u,v} > 0$. With edge $e \in E$ we associate the weight $w_e = w_{u,v}$.

In this chapter, we develop the relationship between random walks on G and electrical networks on G. There are some excellent accounts of this subject area, and the reader is referred to the books of Doyle and Snell [83], Lyons and Peres [221], and Aldous and Fill [19], amongst others. The connection between these two topics is made via the so-called 'harmonic functions' of the random walk.

1.4 Definition Let $U \subseteq V$, and let Z be a Markov chain on V with transition matrix P, that is reversible with respect to the positive function π. The function $f : V \to \mathbb{R}$ is *harmonic* on U (with respect to P) if

$$f(u) = \sum_{v \in V} p_{u,v} f(v), \qquad u \in U,$$

or, equivalently, if $f(u) = \mathbb{E}(f(Z_1) \mid Z_0 = u)$ for $u \in U$.

From the pair (P, π), we can construct the graph G as above, and the weight function w as in (1.2). We refer to the pair (G, w) as the weighted graph associated with (P, π). We shall speak of f as being harmonic (for (G, w)) if it is harmonic with respect to P.

[1] Accounts of Markov chain theory are found in [148, Chap. 6] and [150, Chap. 12].

The so-called hitting probabilities are basic examples of harmonic functions for the chain Z. Let $U \subseteq V$, $W = V \setminus U$, and $s \in U$. For $u \in V$, let $g(u)$ be the probability that the chain, started at u, hits s before W. That is,

$$g(u) = \mathbb{P}_u(Z_n = s \text{ for some } n < T_W),$$

where

$$T_W = \inf\{n \geq 0 : Z_n \in W\}$$

is the first-passage time to W, and $\mathbb{P}_u(\cdot) = \mathbb{P}(\cdot \mid Z_0 = u)$ denotes the conditional probability measure given that the chain starts at u.

1.5 Theorem *The function g is harmonic on $U \setminus \{s\}$.*

Evidently, $g(s) = 1$, and $g(v) = 0$ for $v \in W$. We speak of these values of g as being the 'boundary conditions' of the harmonic function g. See Exercise 1.13 for the uniqueness of harmonic functions with given boundary conditions.

Proof. This is an elementary exercise using the Markov property. For $u \notin W \cup \{s\}$,

$$\begin{aligned} g(u) &= \sum_{v \in V} p_{u,v} \mathbb{P}_u\big(Z_n = s \text{ for some } n < T_W \,\big|\, Z_1 = v\big) \\ &= \sum_{v \in V} p_{u,v} g(v), \end{aligned}$$

as required. □

1.2 Electrical Networks

Throughout this section, $G = (V, E)$ is a finite graph with neither loops nor multiple edges, and $w : E \to (0, \infty)$ is a weight function on the edges. We shall assume further that G is connected.

We may build an electrical network with diagram G, in which the edge e has conductance w_e (or, equivalently, resistance $1/w_e$). Let $s, t \in V$ be distinct vertices termed *sources*, and write $S = \{s, t\}$ for the *source-set*. Suppose we connect a battery across the pair s, t. It is a physical observation that electrons flow along the wires in the network. The flow is described by the so-called Kirchhoff laws, as follows.

To each edge $e = \langle u, v \rangle$, there are associated (directed) quantities $\phi_{u,v}$ and $i_{u,v}$, called the *potential difference* from u to v, and the *current* from u to v, respectively. These are antisymmetric,

$$\phi_{u,v} = -\phi_{v,u}, \qquad i_{u,v} = -i_{v,u}.$$

1.6 Kirchhoff's potential law The cumulative potential difference around any cycle $v_1, v_2, \dots, v_n, v_{n+1} = v_1$ of G is zero, that is,

$$\sum_{j=1}^{n} \phi_{v_j, v_{j+1}} = 0. \tag{1.7}$$

1.8 Kirchhoff's current law The total current flowing out of any vertex $u \in V$ other than the source-set is zero, that is,

$$\sum_{v \in V} i_{u,v} = 0, \qquad u \neq s, t. \tag{1.9}$$

The relationship between resistance/conductance, potential difference, and current is given by Ohm's law.

1.10 Ohm's law For any edge $e = \langle u, v \rangle$,

$$i_{u,v} = w_e \phi_{u,v}.$$

Kirchhoff's potential law is equivalent to the statement that there exists a function $\phi : V \to \mathbb{R}$, called a *potential function*, such that

$$\phi_{u,v} = \phi(v) - \phi(u), \qquad \langle u, v \rangle \in E.$$

Since ϕ is determined up to an additive constant, we are free to pick the potential of any single vertex. Note our convention that *current flows uphill*: $i_{u,v}$ has the same sign as $\phi_{u,v} = \phi(v) - \phi(u)$.

1.11 Theorem *A potential function is harmonic on the set of all vertices other than the source-set.*

Proof. Let $U = V \setminus \{s, t\}$. By Kirchhoff's current law and Ohm's law,

$$\sum_{v \in V} w_{u,v}[\phi(v) - \phi(u)] = 0, \qquad u \in U,$$

which is to say that

$$\phi(u) = \sum_{v \in V} \frac{w_{u,v}}{W_u} \phi(v), \qquad u \in U,$$

where

$$W_u = \sum_{v \in V} w_{u,v}.$$

That is, ϕ is harmonic on U. □

We can use Ohm's law to express potential differences in terms of currents, and thus the two Kirchhoff laws may be viewed as concerning currents only. Equation (1.7) becomes

$$\sum_{j=1}^{n} \frac{i_{v_j,v_{j+1}}}{w_{\langle v_j,v_{j+1}\rangle}} = 0, \tag{1.12}$$

valid for any cycle $v_1, v_2, \dots, v_n, v_{n+1} = v_1$. With (1.7) written thus, each law is linear in the currents, and the superposition principle follows.

1.13 Theorem (Superposition principle) *If i^1 and i^2 are solutions of the two Kirchhoff laws with the same source-set then so is the sum $i^1 + i^2$.*

Next we introduce the concept of a 'flow' on a graph.

1.14 Definition Let $s, t \in V$, $s \neq t$. An *s/t-flow* j is a vector $j = (j_{u,v} : u, v \in V,\ u \neq v)$, such that:

(a) $j_{u,v} = -j_{v,u}$,
(b) $j_{u,v} = 0$ whenever $u \nsim v$,
(c) for any $u \neq s, t$, we have that $\sum_{v \in V} j_{u,v} = 0$.

The vertices s and t are called the 'source' and 'sink' of an s/t flow, and we usually abbreviate 's/t flow' to 'flow'. For any flow j, we write

$$J_u = \sum_{v \in V} j_{u,v}, \qquad u \in V,$$

noting by (c) above that $J_u = 0$ for $u \neq s, t$. Thus,

$$J_s + J_t = \sum_{u \in V} J_u = \sum_{u,v \in V} j_{u,v} = \tfrac{1}{2} \sum_{u,v \in V} (j_{u,v} + j_{v,u}) = 0.$$

Therefore, $J_s = -J_t$, and we call $|J_s|$ the *size* of the flow j, denoted $|j|$. If $|J_s| = 1$, we call j a *unit flow*. We shall normally take $J_s > 0$, in which case s is the *source* and t is the *sink* of the flow, and we say that j is a flow from s to t.

Note that any solution i to the Kirchhoff laws with source-set $\{s, t\}$ is an s/t flow.

1.15 Theorem *Let i^1 and i^2 be two solutions of the Kirchhoff laws with the same source and sink and equal size. Then $i^1 = i^2$.*

Proof. By the superposition principle, $j = i^1 - i^2$ satisfies the two Kirchhoff laws. Furthermore, under the flow j, no current enters or leaves the system. Therefore, $J_v = 0$ for all $v \in V$. Suppose $j_{u_1,u_2} > 0$ for some edge $\langle u_1, u_2 \rangle$. By the Kirchhoff current law, there exists u_3 such that

$j_{u_2,u_3} > 0$. Since $|V| < \infty$, there exists by iteration a cycle $u_l, u_{l+1}, \dots, u_m, u_{m+1} = u_l$ such that $j_{u_k,u_{k+1}} > 0$ for $k = l, l+1, \dots, m$. By Ohm's law, the corresponding potential function satisfies

$$\phi(u_l) < \phi(u_{l+1}) < \cdots < \phi(u_{m+1}) = \phi(u_l),$$

a contradiction. Therefore, $j_{u,v} = 0$ for all u, v. □

For a given size of input current, and given source s and sink t, there can be no more than one solution to the two Kirchhoff laws, but is there a solution at all? The answer is of course affirmative, and the unique solution can be expressed explicitly in terms of counts of spanning trees.[2] Consider first the special case when $w_e = 1$ for all $e \in E$. Let N be the number of spanning trees of G. For any edge $\langle a, b\rangle$, let $\Pi(s, a, b, t)$ be the property of spanning trees that: the unique s/t path in the tree passes along the edge $\langle a, b\rangle$ in the direction from a to b. Let $\mathcal{N}(s, a, b, t)$ be the set of spanning trees of G with the property $\Pi(s, a, b, t)$, and let $N(s, a, b, t) = |\mathcal{N}(s, a, b, t)|$.

1.16 Theorem *The function*

$$i_{a,b} = \frac{1}{N}\big[N(s, a, b, t) - N(s, b, a, t)\big], \qquad \langle a, b\rangle \in E, \tag{1.17}$$

defines a unit flow from s to t satisfying the Kirchhoff laws.

Let T be a spanning tree of G chosen uniformly at random from the set $\mathcal{T}$ of all such spanning trees. By Theorem 1.16 and the previous discussion, the unique solution to the Kirchhoff laws with source s, sink t, and size 1 is given by

$$i_{a,b} = \mathbb{P}\big(T \text{ has } \Pi(s, a, b, t)\big) - \mathbb{P}\big(T \text{ has } \Pi(s, b, a, t)\big).$$

We shall return to uniform spanning trees in Chapter 2.

We prove Theorem 1.16 next. Exactly the same proof is valid in the case of general conductances w_e. In that case, we define the weight of a spanning tree T as

$$w(T) = \prod_{e \in T} w_e,$$

and we set

$$N^* = \sum_{T \in \mathcal{T}} w(T), \qquad N^*(s, a, b, t) = \sum_{T \text{ with } \Pi(s,a,b,t)} w(T). \tag{1.18}$$

The conclusion of Theorem 1.16 holds in this setting with

$$i_{a,b} = \frac{1}{N^*}\big[N^*(s, a, b, t) - N^*(s, b, a, t)\big], \qquad \langle a, b\rangle \in E.$$

[2]This was discovered in an equivalent form by Kirchhoff in 1847, [188].

Proof of Theorem 1.16. We first check the Kirchhoff current law. In every spanning tree T, there exists a unique vertex b such that the s/t path of T contains the edge $\langle s, b\rangle$, and the path traverses this edge from s to b. Therefore,

$$\sum_{b\in V} N(s,s,b,t) = N, \qquad N(s,b,s,t) = 0 \text{ for } b \in V.$$

By (1.17),

$$\sum_{b\in V} i_{s,b} = 1,$$

and, by a similar argument, $\sum_{b\in V} i_{b,t} = 1$.

Let T be a spanning tree of G. The contribution towards the quantity $i_{a,b}$, made by T, depends on the s/t path π of T and equals

$$\text{(1.19)} \qquad \begin{array}{rl} N^{-1} & \text{if } \pi \text{ passes along } \langle a,b\rangle \text{ from } a \text{ to } b, \\ -N^{-1} & \text{if } \pi \text{ passes along } \langle a,b\rangle \text{ from } b \text{ to } a, \\ 0 & \text{if } \pi \text{ does not contain the edge } \langle a,b\rangle. \end{array}$$

Let $v \in V$, $v \neq s,t$, and write $I_v = \sum_{w\in V} i_{v,w}$. If $v \in \pi$, the contribution of T towards I_v is $N^{-1} - N^{-1} = 0$ since π arrives at v along some edge of the form $\langle a, v\rangle$ and departs from v along some edge of the form $\langle v, b\rangle$. If $v \notin \pi$, then T contributes 0 to I_v. Summing over T, we obtain that $I_v = 0$ for all $v \neq s,t$, as required for the Kirchhoff current law.

We next check the Kirchhoff potential law. Let $v_1, v_2, \dots, v_n, v_{n+1} = v_1$ be a cycle C of G. We shall show that

$$\text{(1.20)} \qquad \sum_{j=1}^{n} i_{v_j,v_{j+1}} = 0,$$

and this will confirm (1.12), on recalling that $w_e = 1$ for all $e \in E$. It is more convenient in this context to work with ‘bushes’ than spanning trees. A *bush* (or, more precisely, an s/t *bush*) is defined to be a forest on V containing exactly two trees, one denoted T_s and containing s, and the other denoted T_t and containing t. We write (T_s, T_t) for this bush. Let $e = \langle a, b\rangle$, and let $\mathcal{B}(s,a,b,t)$ be the set of bushes with $a \in T_s$ and $b \in T_t$. The sets $\mathcal{B}(s,a,b,t)$ and $\mathcal{N}(s,a,b,t)$ are in one–one correspondence, since the addition of e to $B \in \mathcal{B}(s,a,b,t)$ creates a unique member $T = T(B)$ of $\mathcal{N}(s,a,b,t)$, and vice versa.

By (1.19) and the above, a bush $B = (T_s, T_t)$ makes a contribution to $i_{a,b}$ of

$$\begin{cases} N^{-1} & \text{if } B \in \mathcal{B}(s,a,b,t), \\ -N^{-1} & \text{if } B \in \mathcal{B}(s,b,a,t), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, B makes a contribution towards the sum in (1.20) that is equal to $N^{-1}(F_+ - F_-)$, where F_+ (respectively, F_-) is the number of pairs v_j, v_{j+1} of C, $1 \le j \le n$, with $v_j \in T_s$, $v_{j+1} \in T_t$ (respectively, $v_{j+1} \in T_s$, $v_j \in T_t$). Since C is a cycle, we have $F_+ = F_-$, whence each bush contributes 0 to the sum and (1.20) is proved. □

1.3 Flows and Energy

Let $G = (V, E)$ be a connected graph as before. Let $s, t \in V$ be distinct vertices, and let j be an s/t flow. With w_e the conductance of the edge e, the (dissipated) *energy* of j is defined as

$$E(j) = \sum_{e=\langle u,v\rangle \in E} j_{u,v}^2 / w_e = \tfrac{1}{2} \sum_{\substack{u,v \in V \\ u \sim v}} j_{u,v}^2 / w_{\langle u,v\rangle}.$$

The following piece of linear algebra will be useful.

1.21 Proposition *Let $\psi : V \to \mathbb{R}$, and let j be an s/t flow. Then*

$$[\psi(t) - \psi(s)]J_s = \tfrac{1}{2} \sum_{u,v \in V} [\psi(v) - \psi(u)] j_{u,v}.$$

Proof. By the properties of a flow,

$$\begin{aligned} \sum_{u,v \in V} [\psi(v) - \psi(u)] j_{u,v} &= \sum_{v \in V} \psi(v)(-J_v) - \sum_{u \in V} \psi(u) J_u \\ &= -2[\psi(s)J_s + \psi(t)J_t] \\ &= 2[\psi(t) - \psi(s)]J_s, \end{aligned}$$

as required. □

Let ϕ and i satisfy the two Kirchhoff laws. We apply Proposition 1.21 with $\psi = \phi$ and $j = i$ to find by Ohm's law that

$$E(i) = [\phi(t) - \phi(s)]I_s. \tag{1.22}$$

That is, the energy of the true current-flow i from s to t equals the energy dissipated in a (notional) single $\langle s, t\rangle$ edge carrying the same potential difference and total current. The conductance W_{eff} of such an edge would satisfy Ohm's law, that is,

$$I_s = W_{\mathrm{eff}}[\phi(t) - \phi(s)], \tag{1.23}$$

and we define the *effective conductance* W_{eff} by this equation. The effective resistance is

$$R_{\mathrm{eff}} = \frac{1}{W_{\mathrm{eff}}}, \tag{1.24}$$

which, by (1.22) and (1.23), equals $E(i)/I_s^2$. We state this as a lemma.

1.25 Lemma *The effective resistance R_{eff} of the network between vertices s and t equals the dissipated energy when a unit flow passes from s to t.*

It is useful to be able to do calculations. Electrical engineers have devised a variety of formulaic methods for calculating the effective resistance of a network, of which the simplest are the series and parallel laws, illustrated in Figure 1.1.

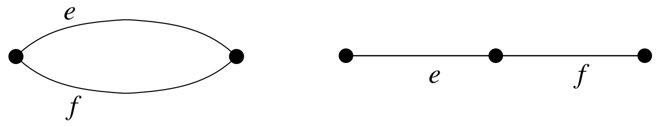

Figure 1.1 Two edges e and f in parallel and in series.

1.26 Series law Two resistors of size r_1 and r_2 in series may be replaced by a single resistor of size $r_1 + r_2$.

1.27 Parallel law Two resistors of size r_1 and r_2 in parallel may be replaced by a single resistor of size R, where $R^{-1} = r_1^{-1} + r_2^{-1}$.

A third such rule, the so-called 'star–triangle transformation', may be found at Exercise 1.5. The following 'variational principle' has many uses.

1.28 Theorem (Thomson principle) *Let $G = (V, E)$ be a connected graph and $(w_e : e \in E)$ strictly positive conductances. Let $s, t \in V$, $s \neq t$. Amongst all unit flows through G from s to t, the flow that satisfies the Kirchhoff laws is the unique s/t flow i that minimizes the dissipated energy. That is,*

$$E(i) = \inf\big\{E(j) : j \text{ a unit flow from } s \text{ to } t\big\}.$$

Proof. Let j be a unit flow from source s to sink t, and set $k = j - i$, where i is the (unique) unit-flow solution to the Kirchhoff laws. Thus, k is a flow with zero size. Now, with $e = \langle u, v\rangle$ and $r_e = 1/w_e$,

$$\begin{aligned} 2E(j) &= \sum_{u,v\in V} j_{u,v}^2 r_e = \sum_{u,v\in V} (k_{u,v} + i_{u,v})^2 r_e \\ &= \sum_{u,v\in V} k_{u,v}^2 r_e + \sum_{u,v\in V} i_{u,v}^2 r_e + 2\sum_{u,v\in V} i_{u,v} k_{u,v} r_e. \end{aligned}$$

Let ϕ be the potential function corresponding to i. By Ohm's law and Proposition 1.21,

$$\begin{aligned} \sum_{u,v\in V} i_{u,v} k_{u,v} r_e &= \sum_{u,v\in V} [\phi(v) - \phi(u)] k_{u,v} \\ &= 2[\phi(t) - \phi(s)] K_s, \end{aligned}$$

which equals zero. Therefore, $E(j) \ge E(i)$, with equality if and only if $j = i$. □

The Thomson 'variational principle' leads to a proof of the 'obvious' fact that the effective resistance of a network is a non-decreasing function of the resistances of individual edges.

1.29 Theorem (Rayleigh principle) *The effective resistance R_{eff} of the network is a non-decreasing function of the edge-resistances $(r_e : e \in E)$.*

It is left as an exercise to show that R_{eff} is a concave function of the vector (r_e). See Exercise 1.6.

Proof. Consider two vectors $(r_e : e \in E)$ and $(r'_e : e \in E)$ of edge-resistances with $r_e \le r'_e$ for all e. Let i and i' denote the corresponding unit flows satisfying the Kirchhoff laws. By Lemma 1.25, with $r_e = r_{\langle u,v\rangle}$,

$$\begin{aligned} R_{\text{eff}} &= \frac{1}{2} \sum_{\substack{u,v\in V \\ u\sim v}} i_{u,v}^2 r_e \\ &\le \frac{1}{2} \sum_{\substack{u,v\in V \\ u\sim v}} (i'_{u,v})^2 r_e && \text{by the Thomson principle} \\ &\le \frac{1}{2} \sum_{\substack{u,v\in V \\ u\sim v}} (i'_{u,v})^2 r'_e && \text{since } r_e \le r'_e \\ &= R'_{\text{eff}}, \end{aligned}$$

as required. □