1
Left Relatively Convex Subgroups

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Abstract

Let G be a group and H be a subgroup of G. We say that H is left relatively convex in G if the left G-set G/H has at least one G-invariant order; when G is left orderable, this holds if and only if H is convex in G under some left ordering of G.

We give a criterion for H to be left relatively convex in G that generalizes a famous theorem of Burns and Hale and has essentially the same proof. We show that all maximal cyclic subgroups are left relatively convex in free groups, in right-angled Artin groups, and in surface groups that are not the Klein-bottle group. The free-group case extends a result of Duncan and Howie.

More generally, every maximal m-generated subgroup in a free group is left relatively convex. The same result is valid, with some exceptions, for compact surface groups. Maximal m-generated abelian subgroups in right-angled Artin groups are left relatively convex.

If G is left orderable, then each free factor of G is left relatively convex in G. More generally, for any graph of groups, if each edge group is left relatively convex in each of its vertex groups, then each vertex group is left relatively convex in the fundamental group; this generalizes a result of Chiswell.

All maximal cyclic subgroups in locally residually torsion-free nilpotent groups are left relatively convex.

1.1 Outline

Notation 1.1 Throughout this chapter, let G be a multiplicative group, and G_0 be a subgroup of G. For x, $y \in G$, $[x, y] := x^{-1}y^{-1}xy$, $x^y := y^{-1}xy$, and ${}^y x := yxy^{-1}$. For any subset X of G, we denote by $X^{\pm 1} := X \cup X^{-1}$, by $\langle X \rangle$ the subgroup of G generated by X, by $\langle X^G \rangle$ the normal subgroup of G generated by X, and let $G/\triangleleft X \triangleright := G/\langle X^G \rangle$. When we write $A \subseteq B$ we mean that A is a subset of B, and when we write $A \subset B$ we mean that A is a proper subset of B.

In Section 1.2, we collect together some facts, several of which first arose in the proof of Theorem 28 of [5]. If G is left orderable, Bergman calls G_0 'left relatively convex in G' if G_0 is convex in G under some left ordering of G, or, equivalently, the left G-set G/G_0 has some G-invariant order. Broadening the scope of his terminology, we shall say that G_0 is *left relatively convex in* G if the left G-set G/G_0 has some G-invariant order, even if G is not left orderable.

We give a criterion for G_0 to be left relatively convex in G that generalizes a famous theorem of Burns and Hale [7] and has essentially the same proof. We deduce that if each noncyclic, finitely generated subgroup of G maps onto $\mathbb{Z}^2$, then each maximal cyclic subgroup of G is left relatively convex in G. Thus, if F is a free group and C is a maximal cyclic subgroup of F, then F/C has an F-invariant order; this extends the result of Duncan and Howie [15] that a certain finite subset of F/C has an order that is respected by the partial F-action. Louder and Wilton [21] used the Duncan–Howie order to prove Wise's conjecture that, for subgroups H and K of a free group F, if H or K is a maximal cyclic subgroup of F, then $\sum_{HxK \in H\backslash F/K} \operatorname{rank}(H^x \cap K) \leq \operatorname{rank}(H)\operatorname{rank}(K)$. They also gave a simple proof of the existence of a Duncan–Howie order; translating their argument from topological to algebraic language led us to the order on F/C. More generally, we introduce the concept of n-indicability and use it to show that each maximal m-generated subgroup of a free group is left relatively convex.

In Section 1.3, we find that the main result of [13] implies that, for any graph of groups, if each edge group is left relatively convex in each of its vertex groups, then each vertex group is left relatively convex in the fundamental group. This generalizes a result of Chiswell [8]. In particular, in a left-orderable group, each free factor is left relatively convex.

One says that G is *discretely left orderable* if some infinite (maximal) cyclic subgroup of G is left relatively convex in G. Many examples of such groups are given in [20]; for instance, it is seen that among free groups, braid groups, surface groups, and right-angled Artin groups, all the infinite ones are discretely left orderable. In Section 1.4, we show that all maximal cyclic subgroups are left relatively convex in right-angled Artin groups and in compact surface groups that are not the Klein-bottle group. More generally, we show that, with some exceptions, each maximal m-generated subgroup of a compact surface group is left relatively convex, and each maximal m-generated abelian subgroup of a right-angled Artin group is left relatively convex.

At the end, in Section 1.5, we show that all maximal cyclic subgroups in locally residually torsion-free nilpotent groups are left relatively convex.

1.2 Left Relatively Convex Subgroups

Definition 1.2 Let X be a set and $\mathcal{R}$ be a binary relation on X; thus, $\mathcal{R}$ is a subset of $X \times X$, and '$x\mathcal{R}y$' means '$(x, y) \in \mathcal{R}$'. We say that $\mathcal{R}$ is *transitive* when, for all x, y, $z \in X$, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$, and here we write $x\mathcal{R}y\mathcal{R}z$ and say that y *fits between* x *and* z *with respect to* $\mathcal{R}$. We say that $\mathcal{R}$ is *trichotomous* when, for all x, $y \in X$, exactly one of $x\mathcal{R}y$, $x = y$, and $y\mathcal{R}x$ holds, and here we say that the *sign* of the triple $(x, \mathcal{R}, y)$, denoted $\operatorname{sign}(x, \mathcal{R}, y)$, is 1, 0, or -1, respectively. A transitive, trichotomous binary relation is called an *order*. For any order $<$ on X, a subset Y of X is said to be *convex in* X *with respect to* $<$ if no element of $X - Y$ fits between two elements of Y with respect to $<$.

Now suppose that X is a left G-set. The diagonal left G-action on $X \times X$ gives a left G-action on the set of binary relations on X. By a *binary* G*-relation on* X we mean a G-invariant binary relation on X, and by a G*-order on* X we mean a G-invariant order on X. If there exists at least one G-order on X, we say that X is G*-orderable*. If X is endowed with a G-order, we say that X is G*-ordered*. When X is G with the left multiplication action, we replace 'G-' with 'left', and write *left order*, *left orderable*, or *left ordered*, the latter two being hyphenated when they premodify a noun.

Analogous terminology applies for right G-sets.

Definition 1.3 For $K \leq H \leq G$, we recall two mutually inverse operations. Let x, $y \in G$.

If $<$ is a G-order on G/K with respect to which H/K is convex in G/K, then we define an H-order $<_{\text{bottom}}$ on H/K and a G-order $<_{\text{top}}$ on G/H as follows. We take $<_{\text{bottom}}$ to be the restriction of $<$ to H/K. We define $xH <_{\text{top}} yH$ to mean $(\forall h_1,\, h_2 \in H)(xh_1K < yh_2K)$. This relation is trichotomous since $xH <_{\text{top}} yH$ holds if and only if we have $(xH \neq yH) \wedge (xK < yK)$; the former clearly implies the latter, and, when the latter holds, $K < x^{-1}yK$, and then, by the convexity of H/K in G/K, $h_1K < x^{-1}yK$, and then $y^{-1}xh_1K < K$, $y^{-1}xh_1K < h_2K$, and $xh_1K < yh_2K$. Thus, $<_{\text{top}}$ is a G-order on G/H.

Conversely, if $<_{\text{bottom}}$ is an H-order on H/K and $<_{\text{top}}$ is a G-order on G/H, we now define a G-order $<$ on G/K with respect to which H/K is convex in G/K. We define $xK < yK$ to mean

$$(xH <_{\text{top}} yH) \;\vee\; \big((xH = yH) \wedge (K <_{\text{bottom}} x^{-1}yK)\big).$$

It is clear that $<$ is a well-defined G-order on G/K. Now suppose that $xK \in (G/K) - (H/K)$. Then $xH \neq H$. If $xH <_{\text{top}} H$, then $xK < hK$, for all $h \in H$, and similarly if $H <_{\text{top}} xH$. Thus, H/K is convex in G/K with respect to $<$.

In particular, G/K has some G-order with respect to which H/K is convex in G/K if and only if H/K is H-orderable and G/H is G-orderable. Taking $K = \{1\}$ and $H = G_0$, we find that the following are equivalent, as seen in the proof of Theorem 28 (vii)⇔(viii) of [5]:

(1.3.1) G has some left order with respect to which G_0 is convex in G,
(1.3.2) G_0 is left orderable, and G/G_0 is G-orderable,
(1.3.3) G is left orderable, and G/G_0 is G-orderable.

This motivates the terminology introduced in the following definition, which presents an analysis similar to one given by Bergman in the proof of Theorem 28 in [5]. Unlike Bergman, we do not require that the group G is left-ordered.

Definition 1.4 Let $\mathrm{Ssg}(G)$ denote the set of all the subsemigroups of G, that is, subsets of G closed under the multiplication. We say that the subgroup G_0 *of* G *is left relatively convex in* G when any of the following equivalent conditions hold:

(1.4.1) the left G-set G/G_0 is G-orderable,
(1.4.2) the right G-set $G_0\backslash G$ is G-orderable,
(1.4.3) there exists some $G_+ \in \mathrm{Ssg}(G)$ such that $G_+^{\pm 1} = G - G_0$; in this event, $G_+ \cap G_+^{-1} = \emptyset$ and $G_0G_+ = G_+G_0 = G_0G_+G_0 = G_+$,

(1.4.4) for each finite subset X of $G{-}G_0$, there exists $S{\in}\mathrm{Ssg}(G)$ such that $X{\subseteq}S^{\pm 1}{\subseteq}G{-}G_0$.

We then say also that G_0 *is a left relatively convex subgroup of* G. One may also use 'right' in place of 'left'.

Proof of equivalence (1.4.1)⇒(1.4.3). Let $<$ be a G-order on G/G_0, and set

$$G_+ := \{x \in G \mid G_0 < xG_0\};$$

then $G_+^{-1} = \{x \in G \mid G_0 < x^{-1}G_0\} = \{x \in G \mid xG_0 < G_0\}$ and $G_0 = \{x \in G \mid G_0 = xG_0\}$. Hence, $G_+^{\pm 1} = G{-}G_0$. If x, $y \in G_+$, then $G_0 < xG_0$, $G_0 < yG_0$ and $G_0 < xG_0 < xyG_0$; thus $xy \in G_+$. Hence, $G_+ \in \mathrm{Ssg}(G)$.

Now consider any $G_+ \in \mathrm{Ssg}(G)$ such that $G_+^{\pm 1} = G{-}G_0$. Then $G_+ \cap G_+^{-1} = \emptyset$, since G_+ is a subsemigroup which does not contain 1. Also, $G_0G_+ \cap G_0 = \emptyset$, since $G_+ \cap G_0^{-1}G_0 = \emptyset$, while $G_0G_+ \cap G_+^{-1} = \emptyset$, since $G_0 \cap G_+^{-1}G_+^{-1} = \emptyset$. Thus $G_0G_+ \subseteq G_+$, and equality must hold. Similarly, $G_+G_0 = G_+$.

(1.4.3)⇒(1.4.1). Let x, y, $z \in G$. We define $xG_0 < yG_0$ to mean that $(xG_0)^{-1}(yG_0) \subseteq G_+$, or, equivalently, that $x^{-1}y \in G_+$. Then $<$ is a well-defined binary G-relation on G/G_0. Since $x^{-1}y$ belongs to exactly one of G_+, G_0, and G_+^{-1}, we see that $<$ is trichotomous. If $xG_0 < yG_0$ and $yG_0 < zG_0$, then G_+ contains $x^{-1}y$, $y^{-1}z$, and their product, which shows that $xG_0 < zG_0$. Thus $<$ is a G-order on G/G_0.

(1.4.2)⇔(1.4.3) is the left-right dual of (1.4.1)⇔(1.4.3).

(1.4.3)⇒(1.4.4) with $S = G_+$.

(1.4.4)⇒(1.4.3). Bergman [5] observes that an implication of this type follows easily from the Compactness Theorem of Model Theory; here, one could equally well use the quasi-compactness of $\{-1,1\}^{G-G_0}$, which holds by a famous theorem of Tychonoff [27]. The case of this implication where $G_0 = \{1\}$ was first stated by Conrad [9], who gave a short argument designed to be read in conjunction with a short argument of Ohnishi [25]. Let us show that a streamlined form of the Conrad–Ohnishi proof gives the general case comparatively easily.

Let 2^{G-G_0} denote the set of all subsets of $G{-}G_0$. For each $W \in 2^{G-G_0}$, let $\mathrm{Fin}(W)$ denote the set of finite subsets of W, and $\langle\langle W\rangle\rangle$ denote the subsemigroup of G generated by W. For each $\varphi \in \{-1,1\}^{G-G_0}$ and $x \in G{-}G_0$, set $\tilde{\varphi}(x) := x^{\varphi(x)} \in \{x, x^{-1}\}$. Set

$$\mathfrak{W} := \Big\{ W \in 2^{G-G_0} \mid (\forall\, W' \in \mathrm{Fin}(W))\ (\forall X \in \mathrm{Fin}(G{-}G_0)) \\ (\exists \varphi \in \{-1,1\}^{G-G_0})\Big(G_0 \cap \langle\langle\, W' \cup \tilde{\varphi}(X)\,\rangle\rangle = \emptyset\Big)\Big\}.$$

It is not difficult to see that (1.4.4) says precisely that $\emptyset \in \mathfrak{W}$. Also, it is clear that

$$(\forall\, W \in 2^{G-G_0})\Big(\big(W \in \mathfrak{W}\big) \Leftrightarrow \big(\mathrm{Fin}(W) \subseteq \mathfrak{W}\big)\Big).$$

It follows that $\mathfrak{W}$ is closed in 2^{G-G_0} under the operation of taking unions of chains. By Zorn's Lemma, there exists some maximal element W of $\mathfrak{W}$.

We shall prove that $\langle\langle W\rangle\rangle^{\pm 1} = G{-}G_0$, and thus (1.4.3) holds. By taking $X = \emptyset$ in the definition of '$W \in \mathfrak{W}$', we see that $\langle\langle W\rangle\rangle \subseteq G{-}G_0$, and thus $W^{\pm 1} \subseteq \langle\langle W\rangle\rangle^{\pm 1} \subseteq G{-}G_0$. It remains to show that $G{-}G_0 \subseteq W^{\pm 1}$. Since W is maximal in $\mathfrak{W}$, it suffices to show that

$$(\forall x \in G{-}G_0)\big((W \cup \{x\} \in \mathfrak{W}) \vee (W \cup \{x^{-1}\} \in \mathfrak{W})\big).$$

Suppose then $W \cup \{x\} \notin \mathfrak{W}$. Thus, we may fix a $W_x \in \mathrm{Fin}(W)$ and an $X_x \in \mathrm{Fin}(G{-}G_0)$ such that

$$\big(\forall \varphi \in \{-1,1\}^{G-G_0}\big)\ \Big(G_0\ \cap\ \big\langle\big\langle\ W_x \cup \{x\} \cup \tilde{\varphi}(X_x)\ \big\rangle\big\rangle \neq \emptyset\Big).$$

Let $W' \in \mathrm{Fin}(W)$ and $X \in \mathrm{Fin}(G{-}G_0)$. As $W \in \mathfrak{W}$, there exists a function $\varphi \in \{-1,1\}^{G-G_0}$ such that

$$G_0 \cap \big\langle\big\langle\ W_x \cup W' \cup \tilde{\varphi}(\{x\} \cup X_x \cup X)\ \big\rangle\big\rangle = \emptyset.$$

Clearly, $\tilde{\varphi}(x) \neq x$. Thus, $\tilde{\varphi}(x) = x^{-1}$ and

$$G_0 \cap \big\langle\big\langle\ W' \cup \{x^{-1}\} \cup \tilde{\varphi}(X)\ \big\rangle\big\rangle = \emptyset.$$

This shows that $W \cup \{x^{-1}\} \in \mathfrak{W}$, as desired. □

The Burns–Hale theorem [7, Theorem 2] says that if each nontrivial, finitely generated subgroup of G maps onto some nontrivial, left-orderable group, then G is left orderable. The following result, using a streamlined version of their proof, generalizes the Burns–Hale theorem in two ways. Namely, the scope is increased by stating the result for an arbitrary subgroup G_0 (in their case G_0 is trivial) and by imposing a weaker condition (in their case $\langle X\rangle$ is required to map onto a left-orderable group).

Theorem 1.5 *If, for each nonempty, finite subset X of $G - G_0$, there exists a proper, left relatively convex subgroup of $\langle X\rangle$ that includes $\langle X\rangle \cap G_0$, then G_0 is left relatively convex in G.*

Proof For each finite subset X of $G{-}G_0$, we shall construct an element $S_X \in \mathrm{Ssg}(\langle X\rangle)$ such that $X \subseteq S_X^{\pm 1} \subseteq G{-}G_0$, and then (1.4.4) will hold. We set $S_\emptyset := \emptyset$. We now assume that $X \neq \emptyset$. Let us write $H := \langle X\rangle$. By hypothesis, we have an H_0 such that $H \cap G_0 \leq H_0 < H$ and H_0 is left relatively convex in H. Notice that $H{-}H_0 \subseteq H{-}(H \cap G_0) \subseteq G{-}G_0$ and $X \cap H_0 \subset X$, since $X \not\subseteq H_0$. By induction on $|X|$, we can find an $S_{X\cap H_0} \in \mathrm{Ssg}(\langle X \cap H_0\rangle)$ such that $X \cap H_0 \subseteq S^{\pm 1}_{X\cap H_0} \subseteq G{-}G_0$. By (1.4.3), since H_0 is left relatively convex in H, we have an $H_+ \in \mathrm{Ssg}(H)$ such that $H_0H_+H_0 = H_+$ and $H_+^{\pm 1} = H - H_0$. We set $S_X := S_{X\cap H_0} \cup H_+$. Then $S_X \in \mathrm{Ssg}(H)$, since $S_{X\cap H_0} \subseteq H_0$ and $H_0H_+H_0 = H_+$. Also,

$$X = (X \cap H_0) \cup (X - H_0) \subseteq S_{X\cap H_0}^{\pm 1} \cup (H - H_0) = S_X^{\pm 1} \subseteq G - G_0. \quad \square$$

Remark Theorem 1.5 has a variety of corollaries. For example, for any subset X of G, we have a sequence of successively weaker conditions: $\langle X \cup G_0\rangle / \lhd G_0 \rhd$ maps onto $\mathbb{Z}$; $\langle X \cup G_0\rangle / \lhd G_0 \rhd$ maps onto a nontrivial, left-orderable group; there exists a proper, left relatively convex subgroup of $\langle X \cup G_0\rangle$ that includes G_0; and, there exists a proper, left relatively convex subgroup of $\langle X\rangle$ that includes $\langle X\rangle \cap G_0$. The last implication follows from the following fact. If A and B are subgroups of G and A is left relatively convex in G, then $A \cap B$ is left relatively convex in B.

Definition 1.6 A group G is said to be *n-indicable*, where n is a positive integer, if it can be generated by fewer than n elements or it admits a surjective homomorphism onto $\mathbb{Z}^n$.

A group G is *locally n-indicable* if every finitely generated subgroup of G is n-indicable.

Note that some authors require in the definition of indicability that G admits a surjective homomorphism onto $\mathbb{Z}$, while here 1-indicable means that G is trivial or maps onto $\mathbb{Z}$, 2-indicable means that G is cyclic or maps onto $\mathbb{Z}^2$, and so on.

Example 1.7 Free abelian groups of any rank and free groups of any rank are locally n-indicable for every n.

The notion of n-indicability is related to left relative convexity through the following corollary of Theorem 1.5.

Corollary 1.8 *Let $n \geq 2$. If G is a locally n-indicable group then each maximal $(n-1)$-generated subgroup of G is left relatively convex in G. In particular, in a free group, each maximal cyclic subgroup is left relatively convex.*

Proof If the subgroup G_0 is a maximal $(n-1)$-generated subgroup of G, then, for any nonempty, finite subset X of $G - G_0$, $\langle X \cup G_0\rangle$ maps onto $\mathbb{Z}^n$, and $\langle X \cup G_0\rangle / \lhd G_0 \rhd$ maps onto $\mathbb{Z}$. $\square$

The idea of Corollary 1.8 can be used to show that certain maximal κ-generated abelian subgroups are left relatively convex, where κ is some cardinal.

Definition 1.9 A group G is *nasmof* if it is torsion-free and every nonabelian subgroup of G admits a surjective homomorphism onto $\mathbb{Z}*\mathbb{Z}$.

Example 1.10 The class of nasmof groups contains free and free abelian groups and it is closed under taking subgroups and direct products. Residually nasmof groups are nasmof, and in particular residually free groups are nasmof. Every nasmof group G is 2-locally indicable, and by Corollary 1.8, maximal cyclic subgroups are left relatively convex.

Corollary 1.11 *Let κ be a cardinal. If G is a nasmof group then each maximal κ-generated abelian subgroup of G is left relatively convex in G.*

In particular, in a residually free group, each maximal κ-generated abelian subgroup is left relatively convex.

Proof Let G_0 be a maximal κ-generated abelian subgroup of G and X a nonempty finite subset of $G - G_0$. By maximality, if $\langle X \cup G_0\rangle$ is abelian, then it is not κ-generated and κ must be a finite cardinal. In this case, $\langle X \cup G_0\rangle$ is a finitely generated, torsion-free abelian group of rank greater than κ. If $\langle X \cup G_0\rangle$ is nonabelian, then it maps onto $\mathbb{Z} * \mathbb{Z}$. In both cases, $\langle X \cup G_0\rangle / \triangleleft G_0 \triangleright$ maps onto $\mathbb{Z}$. □

1.3 Graphs of Groups

Definition 1.12 By a *graph*, we mean a quadruple (Γ, V, ι, τ) such that Γ is a set, V is a subset of Γ, and ι and τ are maps from $\Gamma - V$ to V. Here, we let Γ denote the graph as well as the set, and we write $\mathrm{V}\Gamma := V$ and $\mathrm{E}\,\Gamma := \Gamma - V$, called the *vertex-set* and *edge-set*, respectively. We then define *vertex*, *edge* $\iota e \xrightarrow{e} \tau e$, *inverse edge* $\tau e \xrightarrow{e^{-1}} \iota e$, *path*

(1.12.1) $$v_0 \xrightarrow{e_1^{\epsilon_1}} v_1 \xrightarrow{e_2^{\epsilon_2}} v_2 \xrightarrow{e_3^{\epsilon_3}} \cdots \xrightarrow{e_{n-2}^{\epsilon_{n-2}}} v_{n-2} \xrightarrow{e_{n-1}^{\epsilon_{n-1}}} v_{n-1} \xrightarrow{e_n^{\epsilon_n}} v_n, \quad n \geq 0,$$

reduced path, and *connected graph* in the usual way. We say that Γ is a *tree* if $V \neq \emptyset$ and, for each $(v, w) \in V \times V$, there exists a unique reduced path from v to w. The *barycentric subdivision of* Γ is the graph $\Gamma^{(\prime)}$ such that $\mathrm{V}\Gamma^{(\prime)} = \Gamma$ and $\mathrm{E}\,\Gamma^{(\prime)} = \mathrm{E}\,\Gamma \times \{\iota, \tau\}$, with $e \xrightarrow{(e,\iota)} \iota e$ and $e \xrightarrow{(e,\tau)} \tau e$.

We say that Γ is a *left G-graph* if Γ is a left G-set, V is a G-subset of Γ, and ι and τ are G-maps. For $\gamma \in \Gamma$, we let G_γ denote the G-stabilizer of γ.

Let T be a tree. A *local order on T* is a family $(<_v \mid v \in \mathrm{V}T)$ such that, for each $v \in \mathrm{V}T$, $<_v$ is an order on $\mathrm{link}_T(v) := \{e \in \mathrm{E}T \mid v \in \{\iota e, \tau e\}\}$. By Theorem 3 of [13], for each local order $(<_v \mid v \in \mathrm{V}T)$ on T, there exists a unique order $<_T$ on $\mathrm{V}T$ such that, for each reduced T-path expressed as in (1.12.1),

$$\mathrm{sign}(v_0, <_T, v_n) = \mathrm{sign}\big(0,\ <_{\mathbb{Z}},\ \sum_{i=1}^{n} \epsilon_i + \sum_{i=1}^{n-1} \mathrm{sign}(e_i, <_{v_i}, e_{i+1})\big),$$

where the sign notation is as in Definition 1.2. We then call $<_T$ the *associated order*, $\sum_{i=1}^{n} \epsilon_i$ the *orientation-sum*, and $\sum_{i=1}^{n-1} \mathrm{sign}(e_i, <_{v_i}, e_{i+1})$ the *turn-sum*. If T is a left G-tree, then, for any G-invariant local order on T, the associated order on $\mathrm{V}T$ is easily seen to be a G-order.

Theorem 1.13 *Suppose that T is a left G-tree such that, for each T-edge e, G_e is left relatively convex in $G_{\iota e}$ and in $G_{\tau e}$. Then, for each $t \in T$, G_t is left relatively convex in G. If there exists some $t \in T$ such that G_t is left orderable, then G is left orderable. Moreover, if the input orders are given effectively, then the output orders are given effectively,*

Proof We choose one representative from each G-orbit in $\mathrm{V}T$. For each representative v_0, we choose an arbitrary order on the set of G_{v_0}-orbits $G_{v_0} \backslash \mathrm{link}_T(v_0)$, and, within each G_{v_0}-orbit, we choose one representative e_0 and a G_{v_0}-order on G_{v_0}/G_{e_0}, which exists by (1.4.1); since our G_{v_0}-orbit $G_{v_0}e_0$ may be identified with G_{v_0}/G_{e_0}, we then have a G_{v_0}-order on $G_{v_0}e_0$, and then on all of $\mathrm{link}_T(v_0)$ by our order on $G_{v_0} \backslash \mathrm{link}_T(v_0)$. We then use G-translates to obtain a G-invariant local order on T. This in turn gives the associated G-order on $\mathrm{V}T$ as in Definition 1.12. In particular, for each T-vertex v, we have G-orders on Gv and G/G_v. By (1.4.1), G_v is then left relatively convex in G. For each T-edge e, G_e is left relatively convex in $G_{\iota e}$ by hypothesis, and then G_e is left relatively convex in G by Definition 1.3. Thus, for each $t \in T$, G_t is left relatively convex in G.

By (1.3.2)⇒(1.3.3), if there exists some $t \in T$ such that G_t is left orderable, then G is left orderable. □

Example 1.14 Let F be a free group and X be a free-generating set of F. The left Cayley graph of F with respect to X is a left F-tree on which F acts freely. Thus, the fact that free groups are left orderable can be deduced from Theorem 1.13; see [13].

Bearing in mind that intersections of left relatively convex subgroups are left relatively convex, we can generalize the previous example to the case that a group acts freely on some orbit of n-tuples of elements of T.

Corollary 1.15 *Suppose that T is a left G-tree such that, for each T-edge e, G_e is left relatively convex in $G_{\iota e}$ and in $G_{\tau e}$. Suppose that there exists a finite subset S of T with $\cap_{s \in S} G_s = \{1\}$, then G is left orderable.*

Definition 1.16 By a *graph of groups* $(\mathfrak{G}, \mathit{\Gamma})$, we mean a graph with vertex-set a family of groups $(\mathfrak{G}(v') \mid v' \in \mathrm{V}\,\mathit{\Gamma}^{(')})$ and edge-set a family of injective group homomorphisms $(\mathfrak{G}(e) \xrightarrow{\mathfrak{G}(e')} \mathfrak{G}(v) \mid e \xrightarrow{e'} v \in \mathrm{E}\,\mathit{\Gamma}^{(')})$, where $\mathit{\Gamma}$ is a nonempty, connected graph and $\mathit{\Gamma}^{(')}$ is its barycentric subdivision. For $\gamma \in \mathit{\Gamma}^{(')}$, we call $\mathfrak{G}(\gamma)$ a *vertex group*, *edge group*, or *edge map* if γ belongs to $\mathrm{V}\mathit{\Gamma}$, $\mathrm{E}\,\mathit{\Gamma}$, or $\mathrm{E}\,\mathit{\Gamma}^{(')}$, respectively. One may think of $(\mathfrak{G}, \mathit{\Gamma})$ as a nonempty, connected graph, of groups and injective group homomorphisms, in which every vertex is either a sink, called a vertex group, or a source of valence two, called an edge group. We shall use the *fundamental group* and the *Bass–Serre tree* of $(\mathfrak{G}, \mathit{\Gamma})$ as defined in [26] and [11].

Bass–Serre theory translates Theorem 1.13 into the following form.

Theorem 1.17 *Suppose that G is the fundamental group of a graph of groups $(\mathfrak{G}, \mathit{\Gamma})$ such that the image of each edge map $\mathfrak{G}(e) \xrightarrow{\mathfrak{G}(e')} \mathfrak{G}(v)$ is left relatively convex in its vertex group, $\mathfrak{G}(v)$. Then each vertex group is left relatively convex in G. If some vertex group is left orderable, then G is left orderable. Moreover, if the input orders are given effectively, then the output orders are given effectively.* □

Remark Theorem 1.17 generalizes the result of Chiswell that a group is left orderable if it is the fundamental group of a graph of groups such that each vertex group is left ordered and each edge group is convex in each of its vertex groups; see Corollary 3.5 of [8].

The result of Chiswell is a consequence of Corollary 3.4 of [8], which shows that a group is left orderable if it is the fundamental group of a graph of groups such that each edge group is left orderable and each of its left orders extends to a left order on each of its vertex groups. (If, moreover, each edge group and vertex group is left ordered, and the maps from edge groups to vertex groups respect the orders, then the fundamental group has a left order such that the maps from the vertex groups to the fundamental group respect the orders.) This applies to the case of cyclic edge groups and left-orderable vertex groups.

Corollary 3.4 of [8] is, in turn, a consequence of Chiswell's necessary and sufficient conditions for the fundamental group of a graph of groups