Complex Analysis

Second edition

This new edition of a classic textbook develops complex analysis from the established theory of real analysis by emphasising the differences that arise as a result of the richer geometry of the complex plane. Key features of the authors' approach are to use simple topological ideas to translate visual intuition into rigorous proof, and, in this edition, to address the conceptual conflicts between pure and applied approaches head-on.

Beyond the material of the clarified and corrected original edition, there are three new chapters: Chapter 15 on infinitesimals in real and complex analysis; Chapter 16 on homology versions of Cauchy's Theorem and Cauchy's Residue Theorem, linking back to geometric intuition; and Chapter 17 outlines some more advanced directions in which complex analysis has developed, and continues to evolve into the future.

With numerous worked examples and exercises, clear and direct proofs, and a view to the future of the subject, this is an invaluable companion for any modern complex analysis course.

Ian Stewart, FRS is Emeritus Professor of Mathematics at the University of Warwick. He is author or coauthor of over 190 research papers and is the bestselling author of over 120 books, from research monographs and textbooks to popular science and science fiction. His awards include the Royal Society's Faraday Medal, the IMA Gold Medal, the AAAS Public Understanding of Science Award, the LMS/IMA Zeeman Medal, the Lewis Thomas Prize, and the Euler Book Prize. He is an honorary wizard of the Discworld's Unseen University.

David Tall is Emeritus Professor of Mathematical Thinking at the University of Warwick and is known internationally for his contributions to mathematics education. He is author or coauthor of over 200 papers and 40 books and educational computer software, covering all levels from early childhood to research mathematics.

Complex Analysis

(The Hitch Hiker's Guide to the Plane)

Second edition

IAN STEWART
DAVID TALL
University of Warwick

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108436793

DOI: 10.1017/9781108505468

First published 1983
Thirteenth reprint 2004
Second edition 2018

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication data
Names: Stewart, Ian, 1945– author. | Tall, David Orme, author.
Title: Complex analysis : the hitch hiker's guide to the plane / Ian Stewart
(University of Warwick), David Tall (University of Warwick).
Other titles: Hitch hiker's guide to the plane
Description: Second edition. | Cambridge : Cambridge University Press, 2018.
| Includes bibliographical references and index.
Identifiers: LCCN 2018007009 | ISBN 9781108436793 (alk. paper)
Subjects: LCSH: Functions of complex variables. | Numbers, Complex. |
Geometry, Analytic–Plane.
Classification: LCC QA331 .S85 2018 | DDC 515/.9–dc23
LC record available at https://lccn.loc.gov/2018007009

ISBN 978-1-108-43679-3 Paperback

Additional resources for this publication at www.cambridge.org/Stewart&Tall2ed

Contents

Cambridge University Press & Assessment
978-1-108-43679-3 — Complex Analysis
2nd Edition
Ian Stewart , David Tall
Frontmatter
More Information

Preface to the Second Edition

The first edition of *Complex Analysis* focused on generalising concepts from real analysis to the complex case. Where there were differences, we looked at the geometric picture to see why they were happening. This second edition does the same, but it also focuses on the increasing sophistication of mathematical ideas as we build from intuition to rigour, in a manner where greater understanding leads to more sophisticated intuitions and ways of working. New concepts and methods often start out in a technical way, with problematic aspects that conflict with intuition. As well as generalising real analysis, we move beyond it by addressing these conceptual conflicts, resolving them, and providing more sophisticated concepts and methods appropriate to complex analytic functions.

This approach is used throughout the book. So, for example, the text now includes a short (but complete) discussion of the construction of a space-filling curve, to challenge our intuition about continuity and to explain why we have had to be careful with topological assertions that appear obvious. The treatment here is simpler than most of the literature on space-filling curves. We have spent some time examining different notions of a path, especially the role of smoothness.

We have added three new chapters. Chapter 15 introduces ideas about infinitesimals in real and complex analysis, thought of as variables that tend to zero, and formulated as elements of extensions of the real and complex fields. Chapter 16 gives a formal link from analysis back to geometric intuition, formulating and proving homology versions of Cauchy's Theorem and Cauchy's Residue Theorem. Chapter 17 outlines a few of the more advanced directions in which complex analysis has developed, and continues to evolve into the future.

Chapter 15 has been added for the following reasons. Since the first edition appeared in 1983, the ways in which we operate mathematically have changed dramatically. Not only are there computers that perform numerical and symbolic operations at a speed way beyond that previously available to the individual mind; there are also interactive graphics drawn on high-resolution screens that let us visualise mathematical ideas in completely new ways. In particular, we can dynamically magnify pictures to see tiny detail that lets us represent 'arbitrarily small' quantities.

This second edition therefore includes an extra chapter to introduce formally defined infinitesimals that lie in an ordered extension field K of the real numbers, which can be manipulated algebraically and visualised formally on an extended number line. This approach generalises to the complex case using the field $K(\mathrm{i})$ where $\mathrm{i}^2 = -1$, which can be visualised in the extended complex plane. This construction offers a meaningful bridge between the epsilon-delta rigour of pure mathematics and the intuitive use of infinitesimals in applications.

It can easily be shown that any proper ordered field extension K of the reals must contain infinitesimal elements x: that is, elements that are not zero yet satisfy $|x| < r$ for all positive real numbers r. Using the completeness of the real numbers, we prove a simple theorem that any finite element of K has the form $k = c + h$, where c is real and h is infinitesimal or zero. A transformation in the form $m(x) = (x - c)/\varepsilon$, where ε is a positive infinitesimal, then lets us magnify infinitesimal detail near c and see it with our unaided human eyes in a real picture. This technique extends to the complex case in the field $K(\mathrm{i})$.

We can now illustrate why complex analysis is so different from real analysis. A differentiable complex function defined on an open set is locally expressible as a power series, and we may take K to be the smallest ordered extension field generated by a single infinitesimal ε. The elements are power series $\sum_{r \geq n} a^r \varepsilon^r$ in ε with possibly a finite number of terms in $1/\varepsilon$, and each non-zero element has an order of infinitesimality n related to the first non-zero coefficient a_n (where the element may be infinite if n is negative). Meanwhile a differentiable real function may be differentiable once but not twice, and this requires a much more sophisticated extension field K such as that given by the logical theory of non-standard analysis. While Gottfried Leibniz imagined infinitesimals of different orders, non-standard analysis fails to have this property and requires a much more sophisticated construction. At the end of the chapter we compare and contrast the various theories within a single framework.

Chapter 16 on homology complements Chapter 9 on homotopy versions of Cauchy's Theorem, and logically it could have been placed immediately after that. We postpone it to the penultimate chapter because we do not wish to delay the more practical payoff from Cauchy's Theorem – Taylor and Laurent series, residues, evaluation of integrals, summation of series, and so on.

Homology can be thought of as a way of characterising 'holes' in a topological space, which here is the domain of a complex function f. Singularities, where f is not differentiable, create such holes, and homology helps to describe the topological effect of singularities; for example, in the homology version of Cauchy's Residue Theorem. To avoid including big chunks of algebraic topology, our approach to homology is based on step paths in open subsets of the plane, one of the main simplifying tools in this book. The proof is 'bare hands' and exploits the simple geometry of step paths and the abelian group structure of homology.

Chapter 17 has been included to make it clear that complex analysis is still a major area of mathematical research. Complete though the classical theory may seem to be, there are numerous generalisations and new questions. The main topics mentioned are the Riemann Hypothesis, modular functions, several complex variables, complex manifolds, and complex dynamics – leading to the fractal geometry of Julia sets and the famous Mandelbrot set.

In this new edition of *Complex Analysis* we have corrected all known typographical errors, simplifed some proofs, and reorganised the material in mostly harmless ways to improve readability. We have brought the text and layout into line with current practice, and redrawn all the figures. Proofs, definitions, and examples are terminated with the 'end of proof' symbol □. The same symbol indicates the absence of a proof when the result is clear or has already been proved. Contrary to the prevailing wisdom, we do not insert punctuation marks at the end of displayed formulas. (Your tutors may object to this. Tradition is on their side. If they do, they can set you an extra exercise: *insert all missing punctuation.*) But it is now the twenty-first century. No one puts full stops (US: periods) at the end of book titles, or chapter or section headings. So why do this in displayed formulas, where it may cause confusion because punctuation marks are also often part of the symbolism? We suggest that clean typography should override pedantic punctuation.

Formulas in the main text are another matter; here the *absence* of punctuation can cause confusion. We have followed tradition here.

Online Supplementary Material

Supplementary material including a concordance showing in more detail the changes between the previous edition and this one, and links to *GeoGebra*, can be found on the Cambridge University Press website: www.cambridge.org/Stewart&Tall2ed.

Cambridge University Press & Assessment
978-1-108-43679-3 — Complex Analysis
2nd Edition
Ian Stewart , David Tall
Frontmatter
More Information

Preface to the First Edition

Students faced with a course on 'Complex Analysis' often find it to be just that – complex. In the sense of 'complicated'.

It's true, of course, that the proofs of some of the major theorems in the subject can demand a certain technical versatility. But in many ways, on a conceptual level, complex analysis is actually *easier* than real analysis; it just isn't always taught that way.

This book is intended for use at the level of second or third year undergraduates, and it is based on experience accumulated from teaching such courses over the past decade. To exhibit the inherent simplicity of complex analysis we have organised the material around two basic principles: (1) generalise from the real case, and (2) when that reveals new phenomena, use the rich geometry of the plane to understand them. Our aim throughout is to encourage geometric thinking, with the proviso that it must be adequately backed by analytic rigour.

The opening chapter sets the work in its historical context, and the history is often alluded to later as partial motivation. However, we feel that cultural changes often affect the status of conceptual problems: what was once an important difficulty can become a triviality when viewed with hindsight. It is not always necessary to drag today's students through yesterday's hang-ups. We argue the point at greater length below: it is fundamental to our entire approach.