A Gentle Course in Local Class Field Theory

Local Number Fields, Brauer Groups, Galois Cohomology

This book offers a self-contained exposition of local class field theory, serving as a second course on Galois theory. It opens with a discussion of several fundamental topics in algebra, such as profinite groups, p-adic fields, semisimple algebras and their modules, and homological algebra with the example of group cohomology. The book culminates with the description of the abelian extensions of local number fields, as well as the celebrated Kronecker-Weber theorem, in both the local and global cases. The material will find use across disciplines, including number theory, representation theory, algebraic geometry, and algebraic topology. Written for beginning graduate students and advanced undergraduates, this book can be used in the classroom or for independent study.

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à la mémoire d'Olivier Guillot

Je reconnaissais ce genre de plaisir qui requiert, il est vrai, un certain travail de la pensée sur elle-même, mais à côté duquel les agréments de la nonchalance qui vous font renoncer à lui, semblent bien médiocres. Ce plaisir, dont l'objet n'était que pressenti, que j'avais à créer moi-même, je ne l'éprouvais que de rares fois, mais à chacune d'elles il me semblait que les choses qui s'étaient passées dans l'intervalle n'avaient guère d'importance et qu'en m'attachant à sa seule réalité je pourrais commencer enfin une vraie vie.

Marcel Proust,
À l'ombre des jeunes filles en fleurs

I recognized the kind of pleasure which, admittedly, requires some positive work of the mind upon itself, but compared to which the charms of idleness, that invite you to abandon the effort, seem mediocre. I have felt this pleasure, whose object I could only suspect, and which I had to create myself, only a few times, but it seemed to me that everything which had taken place between these occasions mattered very little, and that I could at last start a true life by clinging to its reality alone.

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Preface

I have taught Galois theory at the undergraduate level for a number of years, with great pleasure. Usually I would follow, with more or less liberty, the first two chapters of Patrick Morandi's book, *Field and Galois theory* [Mor96], taking my favorite detours here and there. At the end of the semester, obviously, I am precisely aware of what the students know and do not know yet, and this is why I have often been embarrassed when asked for advice on choosing a book dealing with Galois theory beyond an introduction. The students I have in mind are not, by a long shot, ready for Jean-Pierre Serre's *Galois cohomology* [Ser02], nor can they start with Serre's *Local fields* [Ser79], to name two classic, beautiful textbooks in the area.

Thus I decided to write an exposition of some topics in Galois cohomology. After much hesitation, I resolved to pick the *Kronecker–Weber theorem* as a final destination, and to include only those facts which are useful for its proof. This celebrated result says that any finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension, a statement which my readers should be able to understand now (see the introduction to Part I for a list of prerequisites). To give another statement that can be appreciated immediately, let me state an easy consequence. Let $P \in \mathbb{Z}[X]$ be a monic polynomial, and assume that the splitting field of P is an *abelian* extension of $\mathbb{Q}$. Then there exists an integer m with the following property: for a prime number p, the question of deciding whether the reduction of P mod p splits into a product of linear factors has an answer that depends only on p mod m (with finitely many exceptions). For $P = X^2 - q$, where q is another prime, one can recover from this the *quadratic reciprocity law*, which says that, in order to decide whether q is a square mod p, you only have to know whether p is a square mod q (and whether p and q are ± 1 mod 4, assuming they are both odd); the generalization is a deep one.

There are several ways to prove the Kronecker–Weber theorem, even if *class field theory*, the theory of abelian extensions of global and local fields (here $\mathbb{Q}$ and its completions are in view), seems inevitable. For example, Childress in [Chi09] gives an account which stays elementary, and is oriented toward students with a strong preference for number theory. We shall follow, by contrast, what is known as the "cohomological approach", here developed from scratch. Let me try to argue in favor of this approach; that it gives me a perfect excuse to include some of my favorite topics should not be seen as its only virtue. My main point is that the various techniques to be discussed will be of interest to many more students and mathematicians than just number theorists. The material in Part I, Part II, and Part III will be useful in many other contexts, and I hope that my readers will find its study rewarding. Let me go through this in more detail, as I give a road map, of sorts, for the book.

- In Part I, we mostly deal with p-adic fields: the various completions $\mathbb{Q}_p$ of $\mathbb{Q}$, and their finite extensions. The field $\mathbb{Q}_p$, and its subring $\mathbb{Z}_p$, should be known to all students who wish to study algebra; and many people do analysis over p-adic fields, too. The p-adic fields form a heaven for Galois theory, their extensions being under very good control (and yet nontrivial): for example, we prove that there are only finitely many extensions $K/\mathbb{Q}_p$ of a given degree, and that $\mathrm{Gal}(K/\mathbb{Q}_p)$ is always a solvable group. Among other preliminaries, we give the basics of topological groups, study briefly vector spaces over complete fields (we discover that $\mathbb{Q}_p$ can replace $\mathbb{R}$ or $\mathbb{C}$ in the classical theorems of analysis), and provide a basic inspection of inverse limits, which appear everywhere in algebra.
- Part II is devoted to *skewfields*, or "noncommutative fields", a topic that is avoided in undergraduate classes, given the maturity that it requires, although students usually ask about the existence of these very early on. As it turns out, the study of skewfields leads us to semisimple algebras, and we end up proving the fundamental results of representation theory over a general field. Of course, complex representations of a finite group can be understood well using characters, but over more complicated fields, semisimple algebras cannot be avoided. We also cover the concept of an extension of a group by another, and explain how these are controlled by a *cohomology group*; this is basic group theory. The first three chapters of Part II can be read independently from Part I, but in the final chapter we fix a p-adic field F and consider the set $\mathrm{Br}(F)$ of all skewfields whose center is precisely F; this set is an abelian group, the *Brauer group* of F, and we prove that it is isomorphic to $\mathbb{Q}/\mathbb{Z}$. This is the first genuinely difficult result in the book.
- Part III deals with *group cohomology*: these are abelian groups written $\mathrm{H}^n(G,M)$, associated with a group G and a G-module M, generalizing the group $\mathrm{H}^2(G,M)$ that appeared in Part II. Collectively, these have astonishing properties. This part is an introduction to the more general phenomena of *homological algebra*, including a discussion of Ext and Tor. Students continuing in algebraic topology or algebraic geometry will face homological algebra all over the place, and group cohomology is a nice first example, on the comparatively concrete side. For students of representation theory, it is quite a revelation to understand that, when one considers a p-group acting on vector spaces of characteristic p, absolutely *nothing* of the usual theory remains useful, and in its place we have the mod p cohomology groups of G. (We shall not explain this connection to representation theory in this book, but we do provide the basic tools which will be needed for it.) Highlights for Part III include Hilbert's Theorem 90, which says that $\mathrm{H}^1(\mathrm{Gal}(K/F),K^\times) = 0$, a fact with many consequences.

- Part IV finally studies class field theory, and proves the Kronecker–Weber theorem. Things are brought together in the following way. We prove Tate's Theorem, which gives a criterion for the existence of isomorphisms of the form $\mathrm{H}^n(G,M) \cong \mathrm{H}^{n-2}(G,\mathbb{Z})$ (in a precise, technical sense). Two ingredients are needed: one related to H^1, and provided by Hilbert's Theorem 90 when $G = \mathrm{Gal}(K/F)$ is a Galois group, and one related to H^2. When looked at the right way, this second, required ingredient turns out to be exactly what Part II was all about. The conclusions of Tate's Theorem can be translated into statements of Galois theory, and are strong enough for us to be able, with some work, to classify all the abelian extensions of p-adic fields – this is called "local class field theory". A "local" version of the Kronecker–Weber theorem follows, about abelian extensions of $\mathbb{Q}_p$. In the final chapter, we go back and forth between number fields (the finite extensions of $\mathbb{Q}$) and their completions, which are p-adic fields, and deduce the "global" Kronecker–Weber theorem. The facts explored in this chapter are the basics of algebraic number theory.

More about the organization of the book can be gathered from the table of contents, and the individual parts have their own introductions, providing some guidance. At the end of each part, references for further reading are given. These include, of course, *Local fields* and *Galois cohomology* by Serre – if I have offered a useful preparation for these masterful expositions, then my work was not in vain. For considerably different reasons, I have also tremendous respect for Blanchard's book, *Les corps non-commutatifs* [Bla72], and for *Cohomology of number fields* [NSW08] by Neukirch, Schmidt, and Wingberg.

The interdependence of chapters is given by Figure 1, in which an arrow from n to m indicates that chapter n must be read in order to understand chapter m. Still, I suggest that you read the chapters from 1 to 14, turning the pages in the usual fashion.

Several people have offered words of encouragement or advice, while this book was being written, and I thank them all warmly. The exposition was, in particular, improved by suggestions of Pierre Baumann, Filippo Nuccio, Chloé Perin, and Olivier Wittenberg. Special thanks are also due to Diana Gillooly at Cambridge University Press for being always so tactful when I had to be convinced to rely on the expertise of others. Finally, anonymous reviewers should know that their work is much appreciated.

Pierre Guillot
Strasbourg, April 2018

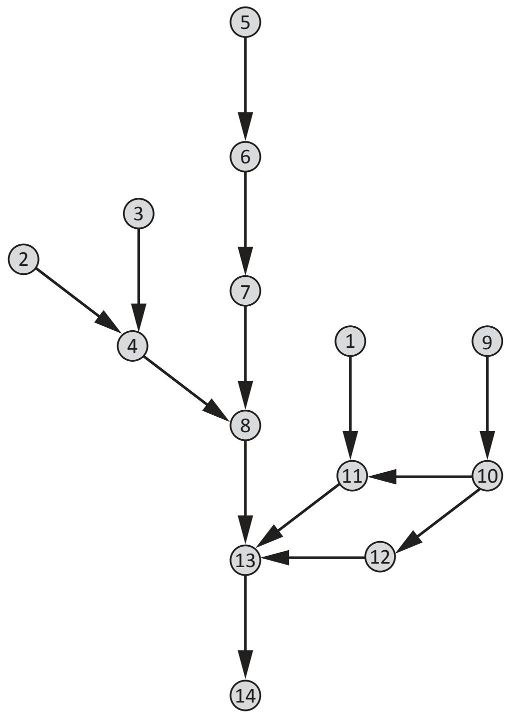

Figure 1. *Leitfaden*