#### A Gentle Course in Local Class Field Theory

Local Number Fields, Brauer Groups, Galois Cohomology

This book offers a self-contained exposition of local class field theory, serving as a second course on Galois theory. It opens with a discussion of several fundamental topics in algebra, such as profinite groups, *p*-adic fields, semisimple algebras and their modules, and homological algebra with the example of group cohomology. The book culminates with the description of the abelian extensions of local number fields, as well as the celebrated Kronecker-Weber theorem, in both the local and global cases. The material will find use across disciplines, including number theory, representation theory, algebraic geometry, and algebraic topology. Written for beginning graduate students and advanced undergraduates, this book can be used in the classroom or for independent study.

PIERRE GUILLOT is a lecturer at the University of Strasbourg and a researcher at the Institut de Recherche Mathématique Avancée (IRMA). He has authored numerous research papers in the areas of algebraic geometry, algebraic topology, quantum algebra, knot theory, combinatorics, the theory of Grothendieck's dessins d'enfants, and Galois cohomology.

# A Gentle Course in Local Class Field Theory

Local Number Fields, Brauer Groups, Galois Cohomology

**Pierre Guillot** University of Strasbourg





Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org Information on this title: www.cambridge.org/9781108432245

DOI: 10.1017/9781108377751

© Pierre Guillot 2018

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press & Assessment.

First published 2018

A catalogue record for this publication is available from the British Library

*Library of Congress Cataloging-in-Publication data* Names: Guillot, Pierre, 1978– author.

Title: A gentle course in local class field theory : local number fields, Brauer groups, Galois cohomology / Pierre Guillot (University of Strasbourg).

Description: Cambridge ; New York, NY : Cambridge University Press, 2019. Identifiers: LCCN 2018026580| ISBN 9781108421775 (hardback : alk. paper) | ISBN 9781108432245 (pbk. : alk. paper)

Subjects: LCSH: Class field theory–Textbooks. | Brauer groups–Textbooks. | Galois theory–Textbooks. | Galois cohomology–Textbooks.

Classification: LCC QA247 .G8287 2019 | DDC 512.7/4-dc23

LC record available at https://lccn.loc.gov/2018026580

ISBN 978-1-108-42177-5 Hardback ISBN 978-1-108-43224-5 Paperback

Cambridge University Press & Assessment has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

à la mémoire d'Olivier Guillot

> Je reconnaissais ce genre de plaisir qui requiert, il est vrai, un certain travail de la pensée sur elle-même, mais à côté duquel les agréments de la nonchalance qui vous font renoncer à lui, semblent bien médiocres. Ce plaisir, dont l'objet n'était que pressenti, que j'avais à créer moi-même, je ne l'éprouvais que de rares fois, mais à chacune d'elles il me semblait que les choses qui s'étaient passées dans l'intervalle n'avaient guère d'importance et qu'en m'attachant à sa seule réalité je pourrais commencer enfin une vraie vie.

> > Marcel Proust, À l'ombre des jeunes filles en fleurs

I recognized the kind of pleasure which, admittedly, requires some positive work of the mind upon itself, but compared to which the charms of idleness, that invite you to abandon the effort, seem mediocre. I have felt this pleasure, whose object I could only suspect, and which I had to create myself, only a few times, but it seemed to me that everything which had taken place between these occasions mattered very little, and that I could at last start a true life by clinging to its reality alone.

#### Contents

	Preface	<i>page</i> xi
	PART I PRELIMINARIES	1
1	Kummer theory	3
	Some basics	3
	Cyclic extensions	5
	Some classical applications	7
	Kummer extensions	8
	The Kummer pairing	10
	The fundamental theorem	14
	Examples	16
2	Local number fields	23
	Construction of $\mathbb{Q}_p$	24
	Around Hensel's lemma	31
	Local number fields	37
	Unramified extensions	43
	Higher ramification groups	49
3	Tools from topology	54
	Topological groups	54
	Profinite groups	57
	Infinite Galois extensions	62
	Locally compact fields	67
4	The multiplicative structure of local number fields	72
	Initial observations	72
	Exponential and logarithm	74
	Module structures	76

CONTENTS

viii

	Summary and first applications	77
	The number of extensions (is finite)	80
	PART II BRAUER GROUPS	83
5	Skewfields, algebras, and modules	85
	Skewfields and algebras	85
	Modules and endomorphisms	89
	Semisimplicity	93
	The radical	98
	Matrix algebras	101
6	Central simple algebras	107
	The Brauer group revisited	107
	Tensor products	108
	The group law	113
	The fundamental theorems	115
	Splitting fields	118
	Separability	120
7	Combinatorial constructions	122
	Introduction	122
	Group extensions	124
	First applications: cyclic groups	130
	Crossed product algebras	134
	Compatibilities	136
	The cohomological Brauer group	141
	Naturality	143
8	The Brauer group of a local number field	146
	Preliminaries	146
	The Hasse invariant of a skewfield	149
	Naturality	153
	PART III GALOIS COHOMOLOGY	157
9	Ext and Tor	159
-	Preliminaries	159
	Resolutions	162
	Complexes	165
	The Ext groups	171
	Long exact sequences	176
	The Tor groups	180

## CAMBRIDGE

	CONTENTS	ix
10	Group cohomology	187
	Definition of group (co)homology	187
	The standard resolution	191
	Low degrees	198
	Profinite groups and Galois cohomology	200
11	Hilbert 90	204
	Hilbert's Theorem 90 in Galois cohomology	204
	More Kummer theory	207
	The Hilbert symbol	208
12	Finer structure	214
	Shapiro's isomorphism	214
	A few explicit formulae	217
	The corestriction	221
	The conjugation action	225
	The five-term exact sequence	227
	Cup-products	230
	Milnor and Bloch–Kato	237
	PART IV CLASS FIELD THEORY	241
13	Local class field theory	243
	Statements	243
	Cohomological triviality and equivalence	244
	The reciprocity isomorphisms	248
	Norm subgroups	250
	Tate duality	252
	The Existence theorem	254
	The local Kronecker–Weber theorem	255
	A concise reformulation	257
14	An introduction to number fields	261
	Number fields and their completions	261
	The discriminant	267
	The (global) Kronecker–Weber theorem	272
	The local and global terminology	275
	Some statements from global class field theory	276
	Appendix: background material	281
	Norms and traces	281
	Tensor products	282
	Notes and further reading	286
	References	290
	Index	292

#### Preface

I have taught Galois theory at the undergraduate level for a number of years, with great pleasure. Usually I would follow, with more or less liberty, the first two chapters of Patrick Morandi's book, *Field and Galois theory* [Mor96], taking my favorite detours here and there. At the end of the semester, obviously, I am precisely aware of what the students know and do not know yet, and this is why I have often been embarrassed when asked for advice on choosing a book dealing with Galois theory beyond an introduction. The students I have in mind are not, by a long shot, ready for Jean-Pierre Serre's *Galois cohomology* [Ser02], nor can they start with Serre's *Local fields* [Ser79], to name two classic, beautiful textbooks in the area.

Thus I decided to write an exposition of some topics in Galois cohomology. After much hesitation, I resolved to pick the *Kronecker–Weber theorem* as a final destination, and to include only those facts which are useful for its proof. This celebrated result says that any finite abelian extension of  $\mathbb{Q}$  is contained in a cyclotomic extension, a statement which my readers should be able to understand now (see the introduction to Part I for a list of prerequisites). To give another statement that can be appreciated immediately, let me state an easy consequence. Let  $P \in \mathbb{Z}[X]$  be a monic polynomial, and assume that the splitting field of P is an *abelian* extension of  $\mathbb{Q}$ . Then there exists an integer m with the following property: for a prime number p, the question of deciding whether the reduction of  $P \mod p$  splits into a product of linear factors has an answer that depends only on  $p \mod m$  (with finitely many exceptions). For  $P = X^2 - q$ , where q is another prime, one can recover from this the *quadratic reciprocity law*, which says that, in order to decide whether q is a square mod p, you only have to know whether p is a square mod q (and whether p and q are  $\pm 1 \mod 4$ , assuming they are both odd); the generalization is a deep one.

There are several ways to prove the Kronecker–Weber theorem, even if *class field theory*, the theory of abelian extensions of global and local fields (here  $\mathbb{Q}$  and its completions are in view), seems inevitable. For example, Childress in [Chi09] gives an account which stays elementary, and is oriented toward students with a strong preference for number theory. We shall follow, by contrast, what is known as the "cohomological approach", here developed from scratch. Let me try to argue in favor

PREFACE

xii

of this approach; that it gives me a perfect excuse to include some of my favorite topics should not be seen as its only virtue. My main point is that the various techniques to be discussed will be of interest to many more students and mathematicians than just number theorists. The material in Part I, Part II, and Part III will be useful in many other contexts, and I hope that my readers will find its study rewarding. Let me go through this in more detail, as I give a road map, of sorts, for the book.

- In Part I, we mostly deal with *p*-adic fields: the various completions  $\mathbb{Q}_p$  of  $\mathbb{Q}$ , and their finite extensions. The field  $\mathbb{Q}_p$ , and its subring  $\mathbb{Z}_p$ , should be known to all students who wish to study algebra; and many people do analysis over *p*-adic fields, too. The *p*-adic fields form a heaven for Galois theory, their extensions being under very good control (and yet nontrivial): for example, we prove that there are only finitely many extensions  $K/\mathbb{Q}_p$  of a given degree, and that  $\operatorname{Gal}(K/\mathbb{Q}_p)$  is always a solvable group. Among other preliminaries, we give the basics of topological groups, study briefly vector spaces over complete fields (we discover that  $\mathbb{Q}_p$  can replace  $\mathbb{R}$  or  $\mathbb{C}$  in the classical theorems of analysis), and provide a basic inspection of inverse limits, which appear everywhere in algebra.
- Part II is devoted to *skewfields*, or "noncommutative fields", a topic that is avoided in undergraduate classes, given the maturity that it requires, although students usually ask about the existence of these very early on. As it turns out, the study of skewfields leads us to semisimple algebras, and we end up proving the fundamental results of representation theory over a general field. Of course, complex representations of a finite group can be understood well using characters, but over more complicated fields, semisimple algebras cannot be avoided. We also cover the concept of an extension of a group by another, and explain how these are controlled by a *cohomology group*; this is basic group theory. The first three chapters of Part II can be read independently from Part I, but in the final chapter we fix a *p*-adic field *F* and consider the set Br(*F*) of all skewfields whose center is precisely *F*; this set is an abelian group, the *Brauer group* of *F*, and we prove that it is isomorphic to Q/Z. This is the first genuinely difficult result in the book.
- Part III deals with *group cohomology*: these are abelian groups written  $H^n(G,M)$ , associated with a group *G* and a *G*-module *M*, generalizing the group  $H^2(G,M)$  that appeared in Part II. Collectively, these have astonishing properties. This part is an introduction to the more general phenomena of *homological algebra*, including a discussion of Ext and Tor. Students continuing in algebraic topology or algebraic geometry will face homological algebra all over the place, and group cohomology is a nice first example, on the comparatively concrete side. For students of representation theory, it is quite a revelation to understand that, when one considers a *p*-group acting on vector spaces of characteristic *p*, absolutely *nothing* of the usual theory remains useful, and in its place we have the mod *p* cohomology groups of *G*. (We shall not explain this connection to representation theory in this book, but we do provide the basic tools which will be needed for it.) Highlights for Part III include Hilbert's Theorem 90, which says that  $H^1(Gal(K/F), K^{\times}) = 0$ , a fact with many consequences.

#### PREFACE

xiii

Part IV finally studies class field theory, and proves the Kronecker–Weber theorem. Things are brought together in the following way. We prove Tate's Theorem, which gives a criterion for the existence of isomorphisms of the form  $H^n(G,M) \cong H^{n-2}(G,\mathbb{Z})$  (in a precise, technical sense). Two ingredients are needed: one related to H<sup>1</sup>, and provided by Hilbert's Theorem 90 when G =Gal(K/F) is a Galois group, and one related to H<sup>2</sup>. When looked at the right way, this second, required ingredient turns out to be exactly what Part II was all about. The conclusions of Tate's Theorem can be translated into statements of Galois theory, and are strong enough for us to be able, with some work, to classify all the abelian extensions of *p*-adic fields – this is called "local class field theory". A "local" version of the Kronecker–Weber theorem follows, about abelian extensions of  $\mathbb{Q}_p$ . In the final chapter, we go back and forth between number fields (the finite extensions of  $\mathbb{Q}$ ) and their completions, which are *p*-adic fields, and deduce the "global" Kronecker–Weber theorem. The facts explored in this chapter are the basics of algebraic number theory.

More about the organization of the book can be gathered from the table of contents, and the individual parts have their own introductions, providing some guidance. At the end of each part, references for further reading are given. These include, of course, *Local fields* and *Galois cohomology* by Serre – if I have offered a useful preparation for these masterful expositions, then my work was not in vain. For considerably different reasons, I have also tremendous respect for Blanchard's book, *Les corps non-commutatifs* [Bla72], and for *Cohomology of number fields* [NSW08] by Neukirch, Schmidt, and Wingberg.

The interdependence of chapters is given by Figure 1, in which an arrow from n to m indicates that chapter n must be read in order to understand chapter m. Still, I suggest that you read the chapters from 1 to 14, turning the pages in the usual fashion.

Several people have offered words of encouragement or advice, while this book was being written, and I thank them all warmly. The exposition was, in particular, improved by suggestions of Pierre Baumann, Filippo Nuccio, Chloé Perin, and Olivier Wittenberg. Special thanks are also due to Diana Gillooly at Cambridge University Press for being always so tactful when I had to be convinced to rely on the expertise of others. Finally, anonymous reviewers should know that their work is much appreciated.

Pierre Guillot Strasbourg, April 2018

## CAMBRIDGE

Cambridge University Press & Assessment 978-1-108-43224-5 — A Gentle Course in Local Class Field Theory Pierre Guillot Frontmatter <u>More Information</u>

xiv

PREFACE



© in this web service Cambridge University Press & Assessment