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Partial Differential Equations arising from Physics and Geometry

A Volume in Memory of Abbas Bahri

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Preface

From March 20 to 29, 2015, a conference bearing the book's name took place in Hammamet, Tunisia.[1]

It was organized by MIMS[2] and CIMPA,[3] and it gave us the opportunity to celebrate the 60th birthday of Professor Abbas Bahri, Rutgers University. Shortly after, Professor Bahri passed away, on January 10, 2016, after a long struggle against sickness. His death caused deep sadness among the academic community and beyond, particularly in Tunisia, France and the United States of America, given the great influence he had in those countries. In Tunisia he created a new school of thought in PDEs, by supervising several students who continue to develop that innovative style. Several memorial tributes took place after his death and many obituaries were published. He will be missed a lot. In this book, we include a chapter presenting a short biography of Professor Bahri, concentrating on his scientific achievements.

Following the Hammamet conference, and given the high quality of the presentations, we felt we should record those contributions by publishing the proceedings of the conference as a book. The majority of the speakers agreed to participate, and we are very grateful to them for their participation in the conference and their commitment to this book.

After the death of Professor Bahri, the book, which was undergoing the refereeing process, suddenly acquired a deeper meaning for all of us, editors and authors: it changed from the status of a simple conference proceedings to that of a tribute to Professor Bahri, dedicated to his memory.

The book's contents reflect, to some extent, the conference talks and courses, which present the state of the art in PDEs, in connection with Professor Bahri's

[1] http://archive.schools.cimpa.info/archivesecoles/20160922162631/
[2] http://www.mims.tn/
[3] https://www.cimpa.info/

contributions. Accordingly, the main speakers at the conference were among the best in their field, mainly from France, the USA, Italy and Tunisia.

MIMS is the Mediterranean Institute for the Mathematical Sciences, founded in Tunis in 2012 to promote mathematics education and research in Tunisia and in the Mediterranean area. It was designed to be a bridge between countries from the North and the South promoting better cooperation.

CIMPA is the International Center for Pure and Applied Mathematics based in Nice, France. It is funded by France, Spain, Switzerland and Norway, together with UNESCO. It promotes mathematical research in developing countries by enhancing North–South cooperation.

Given that many of our contributors are leaders in their field, we expect the book to attract readers from the community of researchers in PDEs interested in interactions with geometry and physics.

We also aim to attract PhD students as readers, since some papers in the book are lecture notes from the six-hour courses given during the conference. We would like to stress the fact that lecturers made the effort of making their courses accessible to PhD students with a basic background in PDEs, as required by CIMPA.

Before closing this preface, we would like to warmly thank again the authors for their valuable contributions. Our thanks go also to Cambridge University Press, for its support with this project, and for carefully considering our submission. We also thank all the production team for handling our LaTeX files with a lot of care and patience. We would also like to acknowledge financial support we received from various institutions, which made the Hammamet conference possible: the Commission for Developing Countries (CDC) of the International Mathematical Union (IMU), the French Embassy in Tunis, University of Carthage, University of Paris 13, University of Tunis El-Manar, University of Sfax, the Tunisian Mathematical Society (SMT) and the Tunisian Association for Applied and Industrial Mathematics (ATMAI).

Paris, June 11, 2017
The editors

Abbas Bahri: A Dedicated Life

This volume is dedicated to the memory of Abbas Bahri. Most of the contributors to this book participated in the conference organized in Hammamet, Tunisia in March 2015, on the occasion of his 60th birthday. A short while later, Abbas passed away on January 10, 2016 after a long illness. In this note, we would like to pay tribute to him, stressing in particular his mathematical achievements and influence.

Abbas Bahri was a leading figure in Nonlinear Analysis and Conformal Geometry. As a matter of fact, he played a fundamental role in our understanding of the lack of compactness arising in some variational problems. For example, his book entitled *Critical Points at Infinity in Some Variational Problems* [3] had a tremendous influence on researchers working in the field of Nonlinear Partial Differential Equations involving critical Sobolev exponents. In particular, he performed in that book the finite-dimensional reduction for Yamabe type problems and the related shadow flow for an appropriate pseudogradient. He also gave the accurate expansion of the Euler–Lagrange functional and its gradient. All these techniques later became widely-used tools in the field.

0.1 A short biography

Abbas Bahri was born on January 1, 1955 in Tunis, Tunisia. At the age of 16 he moved to Paris, where he was admitted to the prestigious École Normale Supérieure, Rue d'Ulm at the age of 19. He later obtained his *Agrégation* in mathematics, then defended a *Thèse d'État* in 1981 at the age of 26 in Université Pierre et Marie Curie (Paris 6), under the direction of Professor Haim Brezis.

Starting his career as a Research Assistant in CNRS between 1979 and 1981, he later obtained other positions in the University of Chicago, École

Polytechnique, Palaiseau, and École Nationale d'Ingénieurs (ENIT), Tunis. In 1988 he was appointed Professor at Rutgers University. As director of the Center for Nonlinear Analysis he organized many seminars and supervised a number of PhD students. He also received many prestigious invitations all over the world. His remarkable achievements have been widely recognized. He was awarded the Langevin and Fermat prizes in 1989 for "introducing new tools in the calculus of variation" and he received in 1990 the Board of Trustees Award for Excellence, Rutgers University's highest honor for outstanding research.

Beyond his mathematical achivements, which will be discussed in the next section, we would like to pay tribute to Abbas Bahri for two other reasons.

The first reason, which is connected to research, concerns his total commitment to "transmission", in particular in his homeland, Tunisia. The decisive act began in the early 1990s, when he started supervising about ten PhD students in Tunisia, including two Mauritanians. He devoted much energy and time to this, dividing his holidays in Tunisia between his family and his students. He was in fact establishing a new "mathematical tradition" in Tunisia, a tradition which is proudly continued by his students who hold many outstanding positions in Tunisia and abroad. More recently, despite his illness, he displayed tremendous courage and went on a "math tour" in Tunisia in 2014–2015, giving lectures at many universities, including École Polytechnique de Tunisie, La Marsa, the University of Kairouan, and the University of Sfax.

The second reason concerns his commitment to progress, democracy, and social justice in the world. He particularly believed in, and fought for, the democratization of his country of origin, where free rational thinking would prevail, and he was confident in the intellectual potential of the Tunisian people.

Besides being a gifted mathematician with an exceptional sense of originality and depth, Abbas Bahri was also interested in – among other things – history, art, music, literature, philosophy, and politics. He believed in the contribution of Arab and Muslim culture to the development of human knowledge and intellect, and as a source of inspiration for progress. He also viewed this contribution as a way to transcend cultural differences. Abbas Bahri valued diversity and nurtured friendships all over the world.

0.2 Mathematical contributions

Abbas Bahri's mathematical interests were very broad, ranging from nonlinear PDEs arising from geometry and physics to systems of differential equations of Celestial Mechanics. However, his research focused mainly on fundamental problems in Contact Forms and Conformal Geometry. Bahri's contributions are various and he published many important results in collaboration with a number of authors.

Bahri was fascinated by variational problems arising in Contact Geometry at the beginning of his career, and he continued to work on this topic for the rest of his life. He was, in particular, motivated by the Weinstein conjecture about the existence of periodic orbits of the Reeb vector field ξ of a given contact form α defined in M^3, a three-dimensional closed and oriented manifold. Although this problem features a variational structure, its corresponding variational form is defined on the loop space of M, $H^1(S^1,M)$, by

$$J(x) := \int_0^1 \alpha(\dot{x})\mathrm{d}t, \quad x \in H^1(S^1,M).$$

In fact, the critical points of J are the periodic orbits of ξ.

J is a very bad variational problem on $H^1(S^1,M)$ because the variational flows do not seem to be Fredholm and the critical points of J have an infinite Morse index.

It is in this framework that Bahri developed the concept of *critical points at infinity*. In fact, he discovered that the ω-limit set of non-compact orbits of the gradient flow behave like a *usual critical point*, once a Morse reduction in the neighborhood of such geometric objects is performed. In particular, one can associate with such *asymptotes* a Morse index as well as stable and unstable manifolds.

To study the functional J, Bahri tried to restrict the variations of the curve. In order to do so, taking a non-vanishing vector field v in $\ker\alpha$ and denoting $\beta(.) := \mathrm{d}\alpha(v,.)$, he defined the subspace $C_\beta := \{x \in H^1(S^1,M) : \beta(\dot{x}) = 0 \text{ and } \alpha(\dot{x}) = a\}$, where a is a positive constant (which may depend on x). Assuming that β is a contact form with the same orientation as α, he proved (in collaboration with D. Bennequin [1]) that:

> J is a C^2 function on C_β whose critical points are the periodic orbits of ξ. Moreover, those orbits have a finite Morse index.

We notice that the curve in C_β can be expressed in a simple way, that is, if $x \in C_\beta$ then $\dot{x} = a\xi + bv$, where a is a positive constant (depending on x) and therefore $J(x) = a > 0$.

It is easy to see that J does not satisfy the Palais–Smale (PS) condition since it just controls the value a of the curve but the b-component along v is free. Therefore, it can have any behavior along a PS sequence.

Crucially, Bahri used the deformation of the level sets of the associated functional J. For this purpose, in general, the used vector field is $-\nabla J$. In the case of the contact form α, taking $\overline{w}$ such that $d\alpha(v,\bar{w}) = 1$, if $z := \lambda\xi + \mu v + \eta\bar{w}$ belongs to T_xC_β (eventually λ, μ and η have to satisfy some conditions (see (2.7) of [8])) then

$$\nabla J(x).z = -\int_0^1 b\eta \mathrm{d}t.$$

In view of this formula, there is a "natural decreasing pseudo-gradient" that can be derived by taking $\eta = b$ in the formula above (the other variables λ and μ will be computed using (2.7) of [8]). This flow has several remarkable geometric properties. One of them is that the linking of two curves under the J-decreasing evolution through the flow of z (with $\eta = b$) never decreases. However, this flow has several "undesired" blow-ups and it is therefore difficult to define a homology related to the periodic orbits of ξ with this pseudo-gradient.

Bahri's main idea was to use a special (constructed) decreasing pseudo-gradient Z for (J, C_β). This program was done in several of his works (in particular [11]) since he required many properties to be satisfied. In particular, the new vector-field Z blows up only along the stratified set $\cup_k\Gamma_{2k}$, where Γ_{2k} := {curves made of $k-$pieces of $\xi-$orbits, alternating with $k-$ pieces of $\pm v-$orbits}. Furthermore, along its (semi)-flow-lines, the number of zeros of b (the v-component of $\dot{x}$) never increases and the L^1-norm of b is bounded. The two points are very important in the study of the PS sequences.

Bahri extended this pseudo-gradient on $\cup\Gamma_{2k}$ and he defined the functional

$$J_\infty(x) := \sum_{k=1}^{\infty} a_k, \quad x \in \cup\Gamma_{2k},$$

where a_k is the length of the kth piece of ξ. The critical points of J_∞ are what Bahri called *critical points at infinity*. This precise pseudo-gradient allowed him to understand the lack of compactness and to characterize the critical points at infinity [11, 9]. These points are characterized as follows. A curve in $\cup\Gamma_{2k}$ is a critical point at infinity if it satisfies one of the following assertions.

1. The v-jumps are between conjugate points (conjugate points are points on the same v-orbit such that the form α is transported onto itself by the transport map along v). These critical points are called *true critical points at infinity*.

2. The ξ-pieces have characteristic length and, in addition, the v-jumps send $\ker\alpha$ to itself (a ξ-piece $[x_0, x_1]$ is characteristic if v completes an exact number $n \in \mathbb{Z}$ of half revolutions from x_0 to x_1).

Furthermore, in [13], Bahri proved that the linking property is conserved, that is: for any decreasing flow-line C_s, originating at a periodic orbit and ending at another periodic orbit O' (contractible in M) of ξ with a difference of indices equal to 1, the linking number $lk(C_s, O')$ never decreases with s.

The properties required for the constructed pseudo-gradient Z allowed Bahri to define an intersection operator ∂ for the variational problem (J, C_β), and he was therefore able to define a kind of homology for the critical points (at infinity) of J [9, 12]. However, he noticed that the critical points (at infinity) do not change the topology of the level sets of J (because J is not Fredholm) and this is a serious difficulty to overcome.

In his last paper [14] Bahri used these properties, combined with the *Fadel–Rabinowitz Morse index*, to present a new beautiful proof of the *Weinstein conjecture* on S^3. This new proof combines the case of the tight contact structure on S^3 and the case of all the over-twisted ones and could therefore lead to a better understanding of the existence process for periodic orbits of ξ. It could also possibly lead to multiplicity results on all three-dimensional closed manifolds with finite fundamental group. Moreover, it can be extended to closed manifolds M^{2n+1} with $n \geq 1$ satisfying some topological assumptions (with some technical difficulties).

When Bahri developed the *theory of critical points at infinity* he applied it to many problems. In collaboration with P. Rabinowitz he studied the 3-body problem in Celestial Mechanics. This problem is modeled by the following Hamiltonian system: $m_i\ddot{q}_i + V_{q_i}(q) = 0$, $i = 1,2,3$, where $m_i > 0$, $q_i \in \mathbb{R}^3$, and $V(q) = \sum_{i\neq j, i,j=1}^{3} m_i m_j/|q_i - q_j|^\alpha$, with $\alpha > 0$. This problem has a variational structure. Its T-periodic solutions correspond to critical points of the functional $I(q) := \int_0^T (\frac{1}{2}\sum m_i|\dot{q}_i|^2 - V(q))dt$ defined on the class of T-periodic functions. In [5] Bahri proved the existence of infinitely many T-periodic solutions of this problem (with $\alpha \geq 2$). The proof of this remarkable result is based on the understanding of the lack of compactness of I. In fact, sections 7 and 8 of [5] are devoted to the analysis of the critical points at infinity of I. This new object allowed the authors to prove the result.

Moreover, Bahri applied his new theory to the Yamabe and scalar curvature problems. For this program, he collected in the monograph [3] many delicate and difficult estimates needed to understand the lack of compactness and to characterize the critical points at infinity of the associated variational functional.

It is known that, in the region where the Palais–Smale condition fails, the functions have to be decomposed as the sum of some bubbles. In collaboration with J.M. Coron, Bahri studied this lack of compactness of the scalar curvature problem on S^3 [4] and he gave a criterion (depending on the scalar function to be prescribed) for the existence of a solution for this problem. This criterion was extended by various authors in other situations and equations with lack of compactness. Furthermore, Bahri used the *theory of critical points at infinity* to give another proof for the Yamabe conjecture for a locally conformally flat manifold [7].

For a bounded domain Ω, the Yamabe problem ($\Delta u + u^{(n+2)/(n-2)}, u > 0$ on Ω; $u = 0$ in $\partial\Omega$) becomes more difficult. The associated variational functional is defined by $J(u) := 1/\int_\Omega |u|^{2n/(n-2)}$ for $u \in \Sigma = \{u \in H_0^1(\Omega) : \|u\| = 1\}$. In collaboration with J.M. Coron, Bahri proved that if Ω has a non-trivial topology then this problem has at least one solution [2]. In fact, the proof is based on combining some analysis and algebraic topology arguments. The analysis part consists of (i) characterizing the levels where the lack of compactness occurs and (ii) proving that there is no difference of topology between the level sets J^a and J^b for a and b large, where $J^a := \{u \in \Sigma : J(u) < a, u > 0\}$. Concerning the algebraic topology argument, since Ω has no trivial topology they were able to find a non-trivial class in the homology of the bottom level set. Furthermore, they proved an intrinsic argument which shows that, for $c_1 < c_2 < c_3$ three consecutive levels (where the lack of compactness occurs), starting from a non-trivial class in the homology of the pair (J^{c_2}, J^{c_1}), if there is no solution, they can find another non-trivial class in the homology of the pair (J^{c_3}, J^{c_2}). Thus, by induction, they are able to find non-trivial classes in the homology of all the pairs $(J^{c_k}, J^{c_{k-1}})$ where the c_is are the levels where the lack of compactness occurs. This gives a contradiction with item (ii) of the analysis part.

Concerning the subcritical case, in collaboration with P.L. Lions, taking a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, Bahri studied the following PDE: (P): $-\Delta u = f(x,u)$ in Ω; $u = 0$ on $\partial\Omega$ with $|f(x,s)| \leq C(1 + |s|^p)$ $(1 < p < (n+2)/(n-2)$ if $n \geq 3)$ and other assumptions on f. In [6], Bahri introduced another type of result. He proved that, taking a sequence of solutions (u_k) of (P), the boundness of $|u_k|_\infty$ is related to the Morse index of (u_k). In fact, the authors proved that the sequence $(|u_k|_\infty)$ is bounded if and only if the sequence of the Morse index of u_k is bounded. The proof relies on some blow-up analysis. The limit problem becomes: $-\Delta u = |u|^{p-1}u$ in $\mathbb{R}^n$ (or $-\Delta u = |u|^{p-1}u$ in Π: a half space with $u = 0$ on $\partial\Pi$). These problems were studied in the positive case, but there is no result for the changing sign solution. Using a bootstrap argument, Bahri proved that these limit problems do not have non-trivial bounded solutions with bounded Morse index. This new criterion (the notion of the Morse index) becomes very useful for classifying the solutions of other equations.

In his last book [10] Bahri studied, in the first part, the changing-sign Yamabe problem. He considered the case of $\mathbb{R}^3$ (or equivalently S^3):

$$\Delta u + u^5 = 0 \qquad \text{in } \mathbb{R}^3. \tag{0.1}$$

In this case the solutions are known to exist, in fact in infinite number. Moreover, if we impose the positivity of the solutions then we see that the only solutions are given by the family $\delta_{(a,\lambda)} := c\sqrt{\lambda}/\sqrt{(1+\lambda^2|.-a|^2)}$. As for changing-sign solutions, we only know their asymptotic behavior at infinity. In this case Bahri studied the asymptotes generated by these solutions and their combinations under the action of the conformal group. As he said in his monograph, *"The Yamabe problem, without the positivity assumption, is a simpler model of less explicit non-compactness phenomena. The equation (0.1) is "un cas d'école""*. In fact, this work provides a family of estimates and techniques by which the problem of finding infinitely many solutions to the changing-sign Yamabe-type problem on domains of $\mathbb{R}^n$, $n \geq 3$, can be studied. Moreover, using the compactness result of Uhlenberg, the ideas introduced in this work could be useful in the study of the Yang–Mills equations. As a matter of fact, this was the topic of the course Bahri gave in February 2015 in the *Faculté des Sciences de Sfax*.

An interesting idea used in [10] consists of deriving an a priori estimate for the remainder term for a PS sequence. Let Ω be a bounded domain and let v be the unique solution of $-\Delta v = f(v)$ on Ω, $v = 0$ in $\partial\Omega$. To prove that $|v(x)| \leq c\varphi(x)$ for each $x \in \Omega$, where φ is a given function, Bahri introduces the following PDE: $-\Delta\widetilde{v} = f(\min(|\widetilde{v}|,\varphi)\text{sign}\,\widetilde{v})$ on Ω, $\widetilde{v} = 0$ in $\partial\Omega$. By studying the new function $\widetilde{v}$, he proves that $|\widetilde{v}|$ is small with respect to φ and therefore it satisfies $-\Delta\widetilde{v} = f(\widetilde{v})$ on Ω, $\widetilde{v} = 0$ in $\partial\Omega$, which implies that $\widetilde{v} = v$ and therefore, v is small with respect to φ. The aim in introducing the new function $\widetilde{v}$ is to overcome some difficulties arising from some non-linear terms of f. Bahri introduced this idea to get some a priori estimate on the remainder function of a PS sequence for the Yamabe sign-changing problem.

As mentioned earlier, Abbas Bahri paved the way for future generations by introducing a new "mathematical tradition" that is now being continued by his students. His contributions go beyond mathematics and his influence has reached many, in Tunisia and all over the world. He is missed not only by his family and friends, but also by many people who met him and appreciated his human qualities and research achievements.

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