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Blow-up Rate for a Semilinear Wave Equation with Exponential Nonlinearity in One Space Dimension

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We consider in this paper blow-up solutions of the semilinear wave equation in one space dimension, with an exponential source term. Assuming that initial data are in $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}$ or sometimes in $W^{1,\infty} \times L^\infty$, we derive the blow-up rate near a non-characteristic point in the smaller space, and give some bounds near other points. Our results generalize those proved by Godin under high regularity assumptions on initial data.

1.1 Introduction

We consider the one dimensional semilinear wave equation:

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + e^u, \\ u(0) = u_0 \text{ and } \partial_t u(0) = u_1, \end{cases} \tag{1.1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}, u_0 \in H^1_{\mathrm{loc},u}$ and $u_1 \in L^2_{\mathrm{loc},u}$. We may also add more restrictions on initial data by assuming that $(u_0, u_1) \in W^{1,\infty} \times L^\infty$. The Cauchy problem for equation (1.1) in the space $H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$ follows from fixed point techniques (see Section 1.2).

If the solution is not global in time, we show in this paper that it blows up (see Theorems 1.1 and 1.2). For that reason, we call it a blow-up solution. The existence of blow-up solutions is guaranteed by ODE techniques and the finite speed of propagation.

More blow-up results can be found in Kichenassamy and Littman [12], [13], where the authors introduce a systematic procedure for reducing nonlinear wave equations to characteristic problems of Fuchsian type and construct

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singular solutions of general semilinear equations which blow up on a non-characteristic surface, provided that the first term of an expansion of such solutions can be found.

The case of the power nonlinearity has been understood completely in a series of papers, in the real case (in one space dimension) by Merle and Zaag [16], [17], [20] and [21] and in Côte and Zaag [6] (see also the note [18]), and in the complex case by Azaiez [3]. Some of those results have been extended to higher dimensions for conformal or subconformal p:

$$1 < p \le p_c \equiv 1 + \frac{4}{N-1}, \tag{1.2}$$

under radial symmetry outside the origin in [19]. For non-radial solutions, we would like to mention [14] and [15] where the blow-up rate was obtained. We also mention the recent contribution of [23] and [22] where the blow-up behavior is given, together with some stability results.

In [5] and [4], Caffarelli and Friedman considered semilinear wave equations with a nonlinearity of power type. If the space dimension N is at most 3, they showed in [5] the existence of solutions of Cauchy problems which blow up on a C^1 spacelike hypersurface. If $N = 1$ and under suitable assumptions, they obtained in [4] a very general result which shows that solutions of Cauchy problems either are global or blow up on a C^1 spacelike curve. In [11] and [10], Godin shows that the solutions of Cauchy problems either are global or blow up on a C^1 spacelike curve for the following mixed problem ($\gamma \neq 1$, $|\gamma| \geq 1$):

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + e^u, \; x > 0, \\ \partial_x u + \gamma \partial_t u = 0 \text{ if } x = 0. \end{cases} \tag{1.3}$$

In [11], Godin gives sharp upper and lower bounds on the blow-up rate for initial data in $C^4 \times C^3$. It so happens that his proof can be extended for initial data $(u_0, u_1) \in H^1_{\text{loc},u} \times L^2_{\text{loc},u}$ (see Proposition 1.15).

Let us consider u a blow-up solution of (1.1). Our aim in this paper is to derive upper and lower estimates on the blow-up rate of $u(x,t)$. In particular, we first give general results (see Theorem 1.1), then, considering only non-characteristic points, we give better estimates in Theorem 1.2.

From Alinhac [1], we define a continuous curve Γ as the graph of a function $x \mapsto T(x)$ such that the domain of definition of u (or the maximal influence domain of u) is

$$D = \{(x,t) | 0 \le t < T(x)\}. \tag{1.4}$$

From the finite speed of propagation, T is a 1-Lipschitz function. The graph Γ is called the blow-up graph of u.

Let us introduce the following non-degeneracy condition for Γ. If we introduce for all $x\in\mathbb{R}$, $t\le T(x)$ and $\delta>0$, the cone

$$\mathcal{C}_{x,t,\delta}=\{(\xi,\tau)\neq(x,t)\,|0\le\tau\le t-\delta|\xi-x|\}, \tag{1.5}$$

then our non-degeneracy condition is the following: x_0 is a non-characteristic point if

$$\exists\delta_0=\delta_0(x_0)\in(0,1) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0}. \tag{1.6}$$

If condition (1.6) is not true, then we call x_0 a characteristic point. We denote by $\mathcal{R}\subset\mathbb{R}$ (resp. $\mathcal{S}\subset\mathbb{R}$) the set of non-characteristic (resp. characteristic) points.

We also introduce for each $a\in\mathbb{R}$ and $T\le T(a)$ the following similarity variables:

$$w_{a,T}(y,s)=u(x,t)+2\log(T-t),\quad y=\frac{x-a}{T-t},\quad s=-\log(T-t). \tag{1.7}$$

If $T=T(a)$, we write w_a instead of $w_{a,T(a)}$.

From equation (1.1), we see that $w_{a,T}$ (or w for simplicity) satisfies, for all $s\ge-\log T$, and $y\in(-1,1)$,

$$\partial_s^2 w-\partial_y((1-y^2)\partial_y w)-e^w+2=-\partial_s w-2y\partial^2_{y,s}w. \tag{1.8}$$

In the new set of variables (y,s), deriving the behavior of u as $t\to T$ is equivalent to studying the behavior of w as $s\to+\infty$.

Our first result gives rough blow-up estimates. Introducing the following set:

$$D_R\equiv\{(x,t)\in(\mathbb{R},\mathbb{R}_+),|x|<R-t\}, \tag{1.9}$$

where $R>0$, we have the following result.

Theorem 1.1 (Blow-up estimates near any point) *We claim the following:*

(i) **(Upper bound)** *For all $R>0$ and $a\in\mathbb{R}$ such that $(a,T(a))\in D_R$, it holds that:*

$$\forall|y|<1,\forall s\ge-\log T(a),\ w_a(y,s)\le-2\log(1-|y|)+C(R),$$
$$\forall t\in[0,T(a)),\ e^{u(a,t)}\le\frac{C(R)}{d((a,t),\Gamma)^2}\le\frac{C(R)}{(T(a)-t)^2},$$

where $d((x,t),\Gamma)$ is the (Euclidean) distance from (x,t) to Γ.

(ii) **(Lower bound)** *For all $R>0$ and $a\in\mathbb{R}$ such that $(a,T(a))\in D_R$, it holds that*

$$\frac{1}{T(a)-t}\int_{I(a,t)}e^{-u(x,t)}dx\le C(R)\sqrt{d((a,t),\Gamma)}\le C(R)\sqrt{T(a)-t}.$$

If, in addition, $(u_0,u_1) \in W^{1,\infty} \times L^\infty$ *then*

$$\forall t \in [0,T(a)), e^{u(a,t)} \geq \frac{C(R)}{d((a,t),\Gamma)} \geq \frac{C(R)}{T(a)-t}.$$

(iii) **(Lower bound on the local energy "norm")** *There exists* $\epsilon_0 > 0$ *such that for all* $a \in \mathbb{R}$*, and* $t \in [0,T(a))$,

$$\frac{1}{T(a)-t}\int_{I(a,t)} ((u_t(x,t))^2 + (u_x(x,t))^2 + e^{u(x,t)})dx \geq \frac{\epsilon_0}{(T(a)-t)^2}, \quad (1.10)$$

where $I(a,t) = (a-(T(a)-t), a+(T(a)-t))$.

Remark The upper bound in item (i) was already proved by Godin [11], for more regular initial data. Here, we show that Godin's strategy works even for less regular data. We refer to the integral in (1.10) as the local energy "norm", since it is like the local energy as in Shatah and Struwe [24], though with the "+" sign in front of the nonlinear term. Note that the lower bound in item (iii) is given by the solution of the associated ODE $u'' = e^u$. However, the lower bound in (ii) doesn't seem to be optimal, since it does not obey the ODE behavior. Indeed, we expect the blow-up for equation (1.1) in the "ODE style", in the sense that the solution is comparable to the solution of the ODE $u'' = e^u$ at blow-up. This is in fact the case with regular data, as shown by Godin [11].

If, in addition, $a \in \mathcal{R}$, we have optimal blow-up estimates.

Theorem 1.2 (An optimal bound on the blow-up rate near a non-characteristic point in a smaller space) *Assume that* $(u_0,u_1) \in W^{1,\infty} \times L^\infty$*. Then, for all* $R > 0$*, for any* $a \in \mathcal{R}$ *such that* $(a,T(a)) \in D_R$*, we have the following:*

(i) **(Uniform bounds on** w**)** *For all* $s \geq -\log T(a) + 1$,

$$|w_a(y,s)| + \int_{-1}^{1} \left((\partial_s w_a(y,s))^2 + (\partial_y w_a(y,s))^2\right) dy \leq C(R),$$

where w_a *is defined in (1.7).*

(ii) **(Uniform bounds on** u**)** *For all* $t \in [0,T(a))$,

$$|u(x,t) + 2\log(T(a)-t)| + (T(a)-t)\int_I (\partial_x u(x,t))^2 + (\partial_t u(x,t))^2 \, dx \leq C(R).$$

In particular, we have

$$\frac{1}{C(R)} \leq e^{u(x,t)}(T(a)-t)^2 \leq C(R).$$

Remark This result implies that the solution indeed blows up on the curve Γ.

Remark Note that when $a \in \mathcal{R}$, Theorem 1.1 already holds and directly follows from Theorem 1.2. Accordingly, Theorem 1.1 is completely meaningful when $a \in \mathcal{S}$.

Following Antonini, Merle and Zaag in [2] and [15], we would like to mention the existence of a Lyapunov functional in similarity variables. More precisely, let us define

$$E(w(s)) = \int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(1-y^2)(\partial_y w)^2 - e^w + 2w \right) dy. \tag{1.11}$$

We claim that the functional E defined by (1.11) is a decreasing function of time for solutions of (1.8) on $(-1,1)$.

Proposition 1.3 (A Lyapunov functional for equation (1.1)) *For all* $a \in \mathbb{R}$, $T \leq T(a)$, $s_2 \geq s_1 \geq -\log T$, *the following identities hold for* $w = w_{a,T}$:

$$E(w(s_2)) - E(w(s_1)) = -\int_{s_1}^{s_2} (\partial_s w(-1,s))^2 + (\partial_s w(1,s))^2 ds.$$

Remark The existence of such an energy in the context of the nonlinear heat equation has been introduced by Giga and Kohn in [7], [8] and [9].

Remark As for the semilinear wave equation with conformal power nonlinearity, the dissipation of the energy $E(w)$ degenerates to the boundary ± 1.

This paper is organized as follows:

In Section 1.2, we solve the local in time Cauchy problem.

Section 1.3 is devoted to some energy estimates.

In Section 1.4, we give and prove upper and lower bounds, following the strategy of Godin [11].

Finally, Section 1.5 is devoted to the proofs of Theorem 1.1, Theorem 1.2 and Proposition 1.3.

1.2 The Local Cauchy Problem

In this section, we solve the local Cauchy problem associated with (1.1) in the space $H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$. In order to do so, we will proceed in three steps.

(1) In Step 1, we solve the problem in $H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$, for some uniform $T > 0$ small enough.
(2) In Step 2, we consider $x_0 \in \mathbb{R}$, and use Step 1 and a truncation to find a local solution defined in some cone $\mathcal{C}_{x_0,\tilde{T}(x_0),1}$ for some $\tilde{T}(x_0) > 0$. Then,

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by a covering argument, the maximal domain of definition is given by $D = \cup_{x_0\in\mathbb{R}} \mathcal{C}_{x_0,\tilde{T}(x_0),1}$.

(3) In Step 3, we consider some approximation of equation (1.1), and discuss the convergence of the approximating sequence.

Step 1: The Cauchy problem in $H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$

In this step, we will solve the local Cauchy problem associated with (1.1) in the space $H = H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$. In order to do so, we will apply a fixed point technique. We first introduce the wave group in one space dimension:

$$S(t) : H \to H,$$

$$(u_0,u_1) \mapsto S(t)(u_0,u_1)(x),$$

$$S(t)(u_0,u_1)(x) = \begin{pmatrix} \frac{1}{2}(u_0(x+t)+u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1 \mathrm{d}t \\ \frac{1}{2}(u_0'(x+t)-u_0'(x-t)) + \frac{1}{2}(u_1(x+t)+u_1(x-t)) \end{pmatrix}.$$

Clearly, $S(t)$ is well defined in H, for all $t \in \mathbb{R}$, and more precisely, there is a universal constant C_0 such that

$$||S(t)(u_0,u_1)||_H \le C_0(1+t)||(u_0,u_1)||_H. \tag{1.12}$$

This is the aim of the step.

Lemma 1.4 (Cauchy problem in $H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$) *For all $(u_0,u_1) \in H$, there exists $T > 0$ such that there exists a unique solution of the problem (1.1) in $C([0,T],H)$.*

Proof Consider $T > 0$ (to be chosen later) small enough in terms of $||(u_0,u_1)||_H$.

We first write the Duhamel formulation for our equation:

$$u(t) = S(t)(u_0,u_1) + \int_0^t S(t-\tau)(0,\mathrm{e}^{u(\tau)})\mathrm{d}\tau. \tag{1.13}$$

Introducing

$$R = 2C_0(1+T)||(u_0,u_1)||_H, \tag{1.14}$$

we will work in the Banach space $E = C([0,T],H)$ equipped with the norm $||u||_E = \sup_{0\le t\le T} ||u||_H$. Then, we introduce

$$\Phi : E \to E$$

$$V(t) = \begin{pmatrix} v(t) \\ v_1(t) \end{pmatrix} \mapsto S(t)(u_0,u_1) + \int_0^t S(t-\tau)(0,\mathrm{e}^{v(t)})\mathrm{d}\tau$$

and the ball $B_E(0,R)$.

We will show that for $T > 0$ small enough, Φ has a unique fixed point in $B_E(0,R)$. To do so, we have to check two points:

1. Φ maps $B_E(0,R)$ to itself;
2. Φ is k-Lipschitz with $k < 1$ for T small enough.

- Proof of 1: Let $V = \begin{pmatrix} v \\ v_1 \end{pmatrix} \in B_E(0,R)$; this means that:

$$\forall t \in [0,T],\ v(t) \in H^1_{\mathrm{loc},u}(\mathbb{R}) \subset L^\infty(\mathbb{R})$$

and that

$$||v(t)||_{L^\infty(\mathbb{R})} \le C_* R.$$

Therefore

$$\begin{aligned} ||(0,\mathrm{e}^v)||_E &= \sup_{0\le t\le T} ||\mathrm{e}^{v(t)}||_{L^2_{\mathrm{loc},u}} \\ &\le \mathrm{e}^{C_*R\sqrt{2}}. \end{aligned} \tag{1.15}$$

This means that

$$\forall \tau \in [0,T]\ (0,\mathrm{e}^{v(\tau)}) \in H,$$

hence $S(t-\tau)(0,\mathrm{e}^{v(\tau)})$ is well defined from (1.12) and so is its integral between 0 and t. So Φ is well defined from E to E.

Let us compute $||\Phi(v)||_E$.

Using (1.12), (1.14) and (1.15) we write for all $t \in [0,T]$,

$$\begin{aligned} ||\Phi(v)(t)||_H &\le ||S(t)(u_0,u_1)||_H + \int_0^t ||S(t-\tau)(0,\mathrm{e}^{v(\tau)})||_H \mathrm{d}\tau \\ &\le \frac{R}{2} + \int_0^T C_0(1+T)\sqrt{2}\mathrm{e}^{C_*R}\mathrm{d}\tau \\ &\le \frac{R}{2} + C_0T(1+T)\sqrt{2}\mathrm{e}^{C_*R}. \end{aligned} \tag{1.16}$$

Choosing T small enough so that

$$\frac{R}{2} + C_0T(1+T)\sqrt{2}\mathrm{e}^{C_*R} \le R$$

or

$$T(1+T) \le \frac{R\mathrm{e}^{-C_*R}}{2\sqrt{2}C_0}$$

guarantees that Φ goes from $B_E(0,R)$ to $B_E(0,R)$.

- Proof of 2: Let $V, \bar{V} \in B_E(0,R)$. We have

$$\Phi(V) - \Phi(\bar{V}) = \int_0^T S(t-\tau)(0, \mathrm{e}^{v(t)} - \mathrm{e}^{\bar{v}(t)})\mathrm{d}\tau.$$

Since $||v(t)||_{L^\infty(\mathbb{R})} \le C_* R$ and the same for $||\bar{v}(t)||_{L^\infty(\mathbb{R})}$, we write

$$|\mathrm{e}^{v(\tau)} - \mathrm{e}^{\bar{v}(\tau)}| \le \mathrm{e}^{C_* R}|v(\tau) - \bar{v}(\tau)|,$$

hence

$$\begin{aligned} ||\mathrm{e}^{v(\tau)} - \mathrm{e}^{\bar{v}(\tau)}||_{L^2_{\mathrm{loc},u}} &\le \mathrm{e}^{C_* R}||v(\tau) - \bar{v}(\tau)||_{L^2_{\mathrm{loc},u}} \\ &\le \mathrm{e}^{C_* R}||V - \bar{V}||_E. \end{aligned} \tag{1.17}$$

Applying $S(t-\tau)$ we write from (1.12), for all $0 \le \tau \le t \le T$,

$$\begin{aligned} ||S(t-\tau)(0, \mathrm{e}^{v(\tau)} - \mathrm{e}^{\bar{v}(\tau)})||_H &\le C_0(1+T)||(0, \mathrm{e}^{v(\tau)} - \mathrm{e}^{\bar{v}(\tau)})||_H \\ &\le C_0(1+T)||\mathrm{e}^{v(\tau)} - \mathrm{e}^{\bar{v}(\tau)}||_{L^2_{\mathrm{loc},u}} \\ &\le C_0(1+T)\mathrm{e}^{C_* R}||V - \bar{V}||_E. \end{aligned} \tag{1.18}$$

Integrating, we end up with

$$||\Phi(V) - \Phi(\bar{V})||_E \le C_0 T(1+T)\mathrm{e}^{C_* R}||V - \bar{V}||_E. \tag{1.19}$$

$k = C_0 T(1+T)\mathrm{e}^{C_* R}$ can be made < 1 if T is small.

Conclusion From points 1 and 2, Φ has a unique fixed point $u(t)$ in $B_E(0,R)$. This fixed point is the solution of the Duhamel formulation (1.13) and of our equation (1.1). This concludes the proof of Lemma 1.4. □

Step 2: The Cauchy problem in a larger region

Let $(u_0, u_1) \in H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$ be initial data for the problem (1.1). Using the finite speed of propagation, we will localize the problem and reduces it to the case of initial data in $H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$ already treated in Step 1. For $(x_0, t_0) \in \mathbb{R} \times (0, +\infty)$, we will check the existence of the solution in the cone $\mathcal{C}_{x_0,t_0,1}$. In order to do so, we introduce χ, a C^∞ function with compact support such that $\chi(x) = 1$ if $|x - x_0| < t_0$; let also $(\bar{u}_0, \bar{u}_1) = (u_0\chi, u_1\chi)$ (note that $\bar{u}_0$ and $\bar{u}_1$ depend on (x_0, t_0) but we omit this dependence in the indices for simplicity). So, $(\bar{u}_0, \bar{u}_1) \in H^1_{\mathrm{loc},u} \times L^2_{\mathrm{loc},u}$. From Step 1, if $\bar{u}$ is the corresponding solution of equation (1.1), then, by the finite speed of propagation, $u = \bar{u}$ in the intersection of their domains of definition with the cone $\mathcal{C}_{x_0,t_0,1}$. As $\bar{u}$ is defined for all (x,t) in $\mathbb{R} \times [0,T)$ from Step 1 for some $T = T(x_0,t_0)$, we get the existence of u locally in $\mathcal{C}_{x_0,t_0,1} \cap \mathbb{R} \times [0,T)$. Varying (x_0,t_0) and covering $\mathbb{R} \times (0, +\infty[$ by an infinite number of cones, we prove the existence and the uniqueness of the solution in a union of backward light cones, which is either the whole half-space $\mathbb{R}\times(0,+\infty)$, or the subgraph of a 1-Lipschitz function $x\mapsto T(x)$. We have just proved the following.

Lemma 1.5 (The Cauchy problem in a larger region) *Consider* $(u_0,u_1)\in H^1_{\mathrm{loc},u}\times L^2_{\mathrm{loc},u}$. *Then, there exists a unique solution defined in D, a subdomain of* $\mathbb{R}\times[0,+\infty)$, *such that for any* $(x_0,t_0)\in D$, $(u,\partial_t u)_{(t_0)}\in H^1_{\mathrm{loc}}\times L^2_{\mathrm{loc}}(D_{t_0})$, *with* $D_{t_0}=\{x\in\mathbb{R}|(x,t_0)\in D\}$. *Moreover,*

- *either* $D=\mathbb{R}\times[0,+\infty)$,
- *or* $D=\{(x,t)|0\le t<T(x)\}$ *for some 1-Lipschitz function* $x\mapsto T(x)$.

Step 3: Regular approximations for equation (1.1)
Consider $(u_0,u_1)\in H^1_{\mathrm{loc},u}\times L^2_{\mathrm{loc},u}$, u its solution constructed in Step 2, and assume that it is non-global, hence defined under the graph of a 1-Lipschitz function $x\mapsto T(x)$. Consider for any $n\in\mathbb{N}$ a regularized increasing truncation of F satisfying

$$F_n(u)=\begin{cases} e^u \text{ if } u\le n,\\ e^n \text{ if } u\ge n+1\end{cases} \tag{1.20}$$

and $F_n(u)\le\min(e^u,e^{n+1})$. Consider also a sequence $(u_{0,n},u_{1,n})\in(C^\infty(\mathbb{R}))^2$ such that $(u_{0,n},u_{1,n})\to(u_0,u_1)$ in $H^1\times L^2(-R,R)$ as $n\to\infty$, for any $R>0$.

Then, we consider the problem

$$\begin{cases}\partial_t^2 u_n=\partial_x^2 u_n+F_n(u_n),\\ (u_n(0),\partial_t u_n(0))=(u_{0,n},u_{1,n})\in H^1_{\mathrm{loc},u}\times L^2_{\mathrm{loc},u}.\end{cases} \tag{1.21}$$

Since Steps 1 and 2 clearly extend to locally Lipschitz nonlinearities, we get a unique solution u_n defined in the half-space $\mathbb{R}\times(0,+\infty)$, or in the subgraph of a 1-Lipschitz function. Since $F_n(u)\le e^{n+1}$, for all $u\in\mathbb{R}$, it is easy to see that in fact u_n is defined for all $(x,t)\in\mathbb{R}\times[0,+\infty)$. From the regularity of F_n, $u_{0,n}$ and $u_{1,n}$, it is clear that u_n is a strong solution in $C^2(\mathbb{R},[0,\infty))$. Introducing the following sets:

$$K^+(x,t)=\{(y,s)\in(\mathbb{R},\mathbb{R}_+),|y-x|<s-t\}, \tag{1.22}$$

$$K^-(x,t)=\{(y,s)\in(\mathbb{R},\mathbb{R}_+),|y-x|<t-s\},$$

and

$$K_R^\pm(x,t)=K^\pm(x,t)\cap\overline{D_R}.$$

We claim the following.

Lemma 1.6 (Uniform bounds on variations of u_n in cones) *Consider $R>0$; one can find $C(R)>0$ such that if $(x,t)\in D\cap\overline{D_R}$, then $\forall n\in\mathbb{N}$:*

$$u_n(y,s)\ge u_n(x,t)-C(R),\ \forall (y,s)\in\overline{K_R^+(x,t)},$$
$$u_n(y,s)\le u_n(x,t)+C(R),\ \forall (y,s)\in\overline{K^-(x,t)}.$$

Remark Of course C depends also on initial data, but we omit that dependence, since we never change initial data in this setting. Note that since $(x,t)\in\overline{D_R}$, it follows that $K_R^-(x,t)=K^-(x,t)$.

Proof We will prove the first inequality, the second one can be proved in the same way. For more details see page 74 of [11].

Let $R>0$, consider (x,t) fixed in $D\cap\overline{D_R}$, and (y,s) in $D\cap\overline{K_R^+(x,t)}$. We introduce the following change of variables:

$$\xi=(y-x)-(s-t),\ \eta=-(y-x)-(s-t),\ \bar{u}_n(\xi,\eta)=u_n(y,s). \quad (1.23)$$

From (1.21), we see that $\bar{u}_n$ satisfies:

$$\partial_{\xi\eta}\bar{u}_n(\xi,\eta)=\frac{1}{4}F_n(\bar{u}_n)\ge 0. \quad (1.24)$$

Let $(\bar{\xi},\bar{\eta})$ be the new coordinates of (y,s) in the new set of variables. Note that $\bar{\xi}\le 0$ and $\bar{\eta}\le 0$. We note that there exists $\xi_0\ge 0$ and $\eta_0\ge 0$ such that the points $(\xi_0,\bar{\eta})$ and $(\bar{\xi},\eta_0)$ lie on the horizontal line $\{s=0\}$ and have as original coordinates respectively $(y^*,0)$ and $(\tilde{y},0)$ for some y^* and $\tilde{y}$ in $[-R,R]$. We note also that in the new set of variables, we have:

$$u_n(y,s)-u_n(x,t)=\bar{u}_n(\bar{\xi},\bar{\eta})-\bar{u}_n(0,0)=\bar{u}_n(\bar{\xi},\bar{\eta})-\bar{u}_n(\bar{\xi},0)+\bar{u}_n(\bar{\xi},0)-\bar{u}_n(0,0)$$
$$=-\int_{\bar{\eta}}^{0}\partial_\eta\bar{u}_n(\bar{\xi},\eta)\mathrm{d}\eta-\int_{\bar{\xi}}^{0}\partial_\xi\bar{u}_n(\xi,0)\mathrm{d}\xi. \quad (1.25)$$

From (1.24), $\partial_\eta\bar{u}_n$ is monotonic in ξ. So, for example for $\eta=\bar{\eta}$, as $\bar{\xi}\le 0\le\xi_0$, we have:

$$\partial_\eta\bar{u}_n(\bar{\xi},\bar{\eta})\le\partial_\eta\bar{u}_n(0,\bar{\eta})\le\partial_\eta\bar{u}_n(\xi_0,\bar{\eta}).$$

Similarly, for any $\eta\in(\bar{\eta},0)$, we can bound from above the function $\partial_\eta\bar{u}_n(\bar{\xi},\eta)$ by its value at the point $(\xi^*(\eta),\eta)$, which is the projection of $(\bar{\xi},\eta)$ on the axis $\{s=0\}$ in parallel to the axis ξ (as $\bar{\xi}\le 0\le\xi^*(\eta)$).

In the same way, from (1.24), $\partial_\xi\bar{u}_n$ is monotonic in η. As $\bar{\eta}\le 0\le\eta_0$, we can bound, for $\xi\in(\bar{\xi},0)$, $\partial_\xi\bar{u}_n(\xi,0)$ by its value at the point $(\xi,\eta^*(\xi))$, which is the projection of $(\xi,0)$ on the axis $\{s=0\}$ in parallel to the axis η $(0<\eta^*(\xi))$. So