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General Editors

B. BOLLOBÁS, W. FULTON, F. KIRWAN,  
P. SARNAK, B. SIMON, B. TOTARO

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**214 The Mathieu Groups**

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# The Mathieu Groups

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## Preface

“There are almost as many different constructions of  $M_{24}$  as there have been mathematicians interested in that most remarkable of all finite groups”.<sup>1</sup>

In this book the study of the Mathieu group  $M_{24}$  (and other Mathieu groups it contains) falls within the scope of what E. E. Shult<sup>2</sup> called the Ivanov–Shpectorov theory of geometries. This theory has been developed to construct and identify large sporadic simple groups including the Baby Monster,<sup>3,4</sup> the Fourth Janko Group  $J_4$ <sup>5</sup> and the Monster.<sup>6</sup> The most dramatic outcome of the theory was the proof of the famous  $Y$ -presentation conjecture for the Monster, which for a long time remained unobtainable by use of the other techniques.<sup>7</sup> In the case of  $M_{24}$  the way in which the theory develops can be projected onto the familiar structures of the Steiner system on 24 points and the Golay code, thus presenting a bold illustration of the theory as well as providing a fresh look at familiar, nearly classical structures. I am extremely grateful to Madeleine Whybrow, William Giuliano and the anonymous referees for suggesting thoughtful corrections, clarifications and modifications after reading earlier versions of the book.

<sup>1</sup> J. H. Conway, The Golay Codes and the Mathieu Groups, in *Sphere Packings, Lattices and Groups*, ed. J. Conway and N.J.A Sloane, Springer, New York, 1988, pp. 299–330.

<sup>2</sup> E. E. Shult, lecture notes on coverings of graphs, Kansas State University, 1997.

<sup>3</sup> A. A. Ivanov, *Geometry of Sporadic Groups I*, Cambridge University Press, Cambridge, 1999.

<sup>4</sup> A. A. Ivanov and S. V. Shpectorov, *Geometry of Sporadic Groups II*, Cambridge University Press, Cambridge, 2002

<sup>5</sup> A. A. Ivanov,  $J_4$ , Oxford University Press, Oxford, 2004.

<sup>6</sup> A. A. Ivanov, *The Monster Group and Majorana Involutions*, Cambridge University Press, Cambridge, 2009.

<sup>7</sup> A. A. Ivanov,  $Y$ -groups via transitive extensions, *J. Algebra* **218** (1999), 412–435.

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