Lectures on Quantum Mechanics

Quantum mechanics is one of the principal pillars of modern physics. It also remains a topic of great interest to mathematicians. Since its discovery it has inspired and been inspired by many topics within modern mathematics, including functional analysis and operator algebras, Lie groups, Lie algebras and their representations, principal bundles, distribution theory, and much more.

Written with beginning mathematics graduate students in mind, this book provides a thorough treatment of nonrelativistic quantum mechanics in a style that is leisurely and without the usual theorem–proof grammar of pure mathematics, while remaining mathematically honest. The author takes the time to develop fully the required mathematics and employs a consistent mathematical presentation to clarify the often-confusing notation of physics texts. Along the way the reader encounters several topics requiring more advanced mathematics than found in many discussions of quantum mechanics. This all makes for a fascinating course in how mathematics and physics interact.

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Lectures on Quantum Mechanics

A Primer for Mathematicians

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Preface

> . . . I have written this work in order to learn the subject myself, in a form I find comprehensible. And readers familiar with some of my previous books probably realize that this has pretty much been the reasons for those works also . . . Perhaps this travelogue of an innocent abroad in a very different field will also turn out to be a book that mathematicians will enjoy (though physicists probably will not).
>
> Michael Spivak
> in the Preface to *Physics for Mathematicians: Mechanics I*
> (2010)

Apologia

I imagine many authors write first and foremost for themselves. I expect that, often, writing a textbook on a technical subject is the author's way of mastering the subject matter, of coming to grips with technical difficulties, and of arranging topics of keen interest to himself or herself, certainly with the reader in mind but motivated even more so by his or her own desire for ordering and framing a difficult subject. I must confess guilt in these matters. This book ultimately is written to myself of 40 years ago when I was a graduate student in mathematics with several core graduate courses in pure mathematics under my belt.

I was originally a physics major as an undergraduate. Like many of my colleagues who ended up in pure mathematics, I ran from the messy mathematical imprecision of physics to the elegant, precise, and rigorous garden of pure mathematics. By the time I had acquired the mathematical tools to understand, with the rigor my psychology demanded, what the physicists were doing, I was captured by the beauty and elegance of pure mathematics, and any professional aspirations I had held for a career in physics had morphed into aspirations for one in mathematics. But that keen interest in physics that had been fueled and fanned into flame upon reading George Gamow's *One, Two, Three, Infinity* and the Isaac Asimov books on physics, while in high school, remained. Over the past four decades I have, sometimes with less, sometimes with greater intensity, continually filled in the gaps in my understanding of quantum mechanics, and this book is my report to my 20-year-old self. It is a roadmap that would have made my attempt to gain a precise understanding of the mathematical difficulties of quantum mechanics a bit easier had it sat on my desk four decades ago. It is my hope that this book might be of use to the student of mathematics who is in the same predicament.

The reader I target is that student of mathematics who has perhaps a year of graduate training, who has seen Hilbert spaces or measure, who knows something of manifolds or Lie groups, who has been introduced to tensor products, though much of this might still seem quite mysterious and exotic. I am not looking for familiarity with functional analysis at the level of the spectral theorem or Stone's theorem, nor familiarity of group theory at the level of representation theory – these will be explained when needed – but I am looking for a desire to understand quantum mechanics coupled with just the right amount of mathematical sophistication and preparation that allows for a mathematically honest and thorough, if not quite rigorous, treatment of quantum mechanics. The level of sophistication necessary is at least, but really no more than that of the graduate student of mathematics at the end of the first year of study at a US university.

Quite a bit of modern mathematics that is absent from most physicists' expositions is used, reviewed, or developed in this book. I hope that the student of mathematics will find in it just the right amount of mathematical explanation to satisfy the need for rigor yet a presentation that is not overly pedantic.

The exposition and choice of topics bear the strong imprint of the author, and so the reader will find herein some rather unorthodox topics as well as some rather unorthodox approaches to standard topics, all of which are flavored by the author's personal mathematical prejudices as well as his personal pedagogical perspective. This, I imagine, is true of most works of most authors, and some of the particularly distinctive features of this presentation will be discussed in more detail next.

Some Distinctive Features

This is not a textbook. There are no exercises. It fails to present many of the practical tools – the various approximation methods for example – that physicists use to do calculations, a deficiency for use as a textbook for the student of physics. It repeats material found in a variety of proper mathematics courses, but without the degree of completeness and generality one expects of a good mathematics text, and often describes mathematical results without the benefit of rigorous justification – the mathematics is always honest but not always rigorously justified! Thus, I do not consider this work an exposition of mathematics but rather an explanation of quantum mechanics in as mathematically thorough a way as I know, and I have chosen to present the mathematics as needed as part of the physics story. In particular, I let the physics take center stage and drive the mathematics. For this, prose seems more appropriate than technical mathematical syntax for the telling of this story, and so I have dispensed with the familiar and precise mathematical syntax of axiom–definition–lemma–proposition–theorem–corollary–QED so dear to pure mathematicians. In fact, I try to present much of the material in a style that physics students appreciate but all the while taking great care with the mathematics. Of course this requires some extensive pauses to consider the relevant mathematics, some of which is developed precisely and in detail. This approach already differentiates this work from many that serve first as textbooks of mathematics, with the physics simply the motivation for the mathematics.

As an example of how this book differs from those, we employ the spectral theorem for unbounded operators to make sense of the so-called projection postulate of physicists, but we omit a proof of the spectral theorem even though we do not expect the reader to be familiar with this result; rather, we carefully explain the spectral theorem and how it may be used to encode the physical content of the theory. This should have the effect of reassuring students that the quantum mechanical formalism rests on a sound, rigorous footing and may inspire the more curious and motivated ones to learn the proof on their own, either by taking an appropriate course in functional analysis or by consulting the list of references provided at the end of the book.

Aside from expectations of the reader that are more mathematically advanced than those for most books on quantum mechanics, there are some rather distinctive features of this book in terms of both the choice of topics and their development. These features are detailed in the prolegomenon that follows this preface, but I should like to point out some of them now. Herein the reader may find the following: a rigorous treatment of the measurement axiom and the projection postulate that employs the spectral theorem for unbounded operators; a development of the Dirac calculus, the famous bra-ket notation, in the context of rigged Hilbert spaces; a careful discussion of commuting operators; a detailed review of the Fourier transform for use in the momentum representation; angular momentum operators as $\mathfrak{su}(2)$ representations; the employment of quaternions in both the development of SU(2) representations and in transforming the central force problem in three-dimensional space to the 3-sphere; a detailed discussion of the hidden SO(4) symmetry of the hydrogenic potential using stereographic projection and unit quaternions; Fock's treatment of the momentum space equation as well as a modern treatment using group characters; the use of tensor products to incorporate internal variables, with Pauli's theory of the electron as an example; a development of the Clifford algebra approach to spin; a careful discussion of the Einstein–Podolsky–Rosen paradox and nonlocality, including the Greenberger–Horne–Zeilinger scheme; a careful development of Fock space; the use of Stone's theorem on unitary representations to "derive" the Schrödinger equation and to understand conservation laws; a somewhat novel interpretation of the Feynman approach via path integrals; and a careful exposition of Gaussian integrals.

My hope for the student studying these topics is the same excitement and satisfaction that I found when discovering for myself the great work of the physicists and mathematicians who constructed quantum mechanics during the first half of the last century.

Homage

It is a pleasure to acknowledge three treatises on quantum mechanics that have been instrumental in shaping the general tenor and tone of the present work. These made a great impression on me, not only in my understanding of some key features of the subject but, more importantly, on how to present the subject with one's own personal elegance and élan. The present work may be considered an extended homage to these three books and my fond hope is that it may be seen as a worthy complement to them, though emphasizing different aspects of the subject and from a perspective that differs in some significant ways from theirs.

Eugene Merzbacher published the first edition of his book *Quantum*

Mechanics in 1961. I embarked on my first systematic study of quantum mechanics by a rather sporadic reading of the second edition, published in 1970, a copy of which I had acquired in October 1977. I was delighted when Professor Merzbacher produced a substantially updated third edition in 1998, which I consider to be as definitive an exposition of quantum mechanics from the perspective of a practitioner of the discipline as one can find. The publishing house Wiley has done a great service by keeping this work in print for so many years, a work that I think deserves the term *magisterial*. It is the case even today that I can turn randomly to its pages and find some bit of the subject that has eluded my notice, and I can learn from an expositor so well versed and at home with the physical reality behind the mathematics that his explications seem effortless and inevitable. Merzbacher has a genius for presenting the physics front and center but with enough mathematical honesty to satisfy even this most unrepentant mathematician.

In January 1999 at the Annual American Mathematical Society meeting in San Antonio I had the good fortune of stumbling upon Keith Hannabuss's beautiful text, *An Introduction to Quantum Theory*. This had been published two years earlier as the first work in the series *Oxford Graduate Texts in Mathematics*. If Merzbacher was my introduction to quantum theory, Hannabuss was my guide on how to present the theory with great clarity and elegance by adopting rigorous modern mathematics for its telling. I had read von Neumann's historically important and mathematically sound rendition of the subject in his *Mathematical Foundations of Quantum Mechanics* and was familiar with Chris Isham's beautiful little gem of 1995, *Lectures on Quantum Theory*, a mathematically robust "second course" in the theory written by a famed theoretical physicist with a strong training in pure mathematics, but it was the text by Hannabuss that captured my imagination in its choice of material and style of presentation. I absorbed this book over the following semester and based many lectures on Hannabuss's exposition in a seminar on quantum mechanics that I organized over the next two years. Among the scores of books and other sources from which I have learnt the subject over the past four decades, this book owes its greatest debt to Hannabuss. I consider the present work as a companion to Hannabuss's and would hope that the reader who appreciates his book would find mine quite pleasing to his or her temperament.

Finally I must express my admiration of Shlomo Sternberg's 1994 treatise *Group Theory and Physics*, certainly one of the more unique contributions to the literature of physics and mathematics and one of my favorite books. The ease with which Sternberg mixes the facts of the physical world with sophisticated mathematics and uses the mathematics to provide rigorous explanations and perceptive insights is impressive. He flows seamlessly from a physical description of a problem to a development of the rigorous mathematics needed to attack the problem, and then back to the problem with a beautiful application. He does this while dispensing, like this book, with formal mathematical syntax, and so his explication of the subject unfolds like a good story being told. I am rather captured by Sternberg's style of laying out the story he tells, and the present work in style and temperament is most akin to his. Included as an appendix to his book is a rather wonderful essay on the history of nineteeth century spectroscopy. It is amazing that the seemingly pedestrian act of measuring the characteristic spectral lines of nature's elements leads invariably to an explanation in which group representation theory takes a central role.

Gratitude

I would be remiss were I to fail to acknowledge and thank the folks and institutions that have made it possible for me to write this book. It has been a journey of 45 years. First I must acknowledge the University of North Carolina at Asheville, whose dedicated professors first nurtured the wide-eyed enthusiasm for physics and mathematics they found in a rather naïve freshman in 1974, as well as the University of Tennessee, which brought me to maturity in mathematics, and my doctoral mentor, John J. Walsh, who taught me how to think like a mathematician. To The Florida State University in Tallahassee I owe a great debt of gratitude for providing a stimulating environment over the past 36 years in which to build a career in mathematics, which gave me time to think and learn and do research wherever my interests turned and provided stimulating students and classes to teach. I should mention Cambridge University, venue of my 1996 sabbatical, where my interest in quantum mechanics was renewed after a hiatus of several years. The town of Cambridge provided exceptional bookstores to browse, where I found stimulating books on aspects of quantum mechanics that had escaped my off-and-on study of the subject since my graduate school days.

The two individuals who have had the greatest impact on my mathematical work are Jim Cannon and Ken Stephenson. Jim's work has influenced mine significantly, and I greatly admire his mathematical tastes

and contributions. To collaborate with Ken has been a joy over the past two decades. We have been co-authors on a number of research articles and his down-to-earth approach to the understanding of mathematics has been a constant check on my tendency toward flights of fancy. I have learned from him how to tell a good story about a mathematical topic.

I must extend thanks to all those who attended my lectures over the years, but especially to Ettore Aldrovandi, with whom I often discussed topics in depth, to Paolo Aluffi, one of my chief cheerleaders in this effort and another discussion participant, and to Matilde Marcolli, who listened to some of my far-fetched notions on the Feynman path integral. I must also thank the students in two graduate classes I taught – Quantum Mechanics for Mathematicians, I and II – in the spring and fall semesters of 2011, whose questions helped fine tune much of the presentation of this book. I mention with fondness my former office neighbor, Jack Quine, who shares my enthusiasm for both quantum mechanics and our other guilty pleasure, number theory. Washington Mio and I had a brief flirtation with quantum computing, and he graciously tolerated my intrusions and prodding to speak in seminar about the topic.

It has been a pleasure to work with the wonderful folks at Cambridge University Press in preparing this book for publication. Over a period of several years at annual AMS Joint Mathematics Meetings I spoke with mathematics editor Roger Ashley about a book of this type and he was always a great encouragement. When finally I was able to submit a manuscript, I was placed in the capable hands of mathematics editor Tom Harris and his able assistant Anna Scriven. Tom and Anna have been a source of support and great patience as I juggled my professional responsibilities as a professor at a research university with the technical editing of a book of this length. It has been a joy to work with senior content manager Esther Migueliz Obanos and copy editor Susan Parkinson in this process. Esther was always understanding when I failed to meet suggested deadlines and with gentle prodding directed me towards manageable goals. Susan is a copy editor par excellence whose edits, suggestions, and questions were always spot-on and insightful. Susan has a PhD in spectroscopy and has taught quantum mechanics at the Open University in the UK. Her work greatly improved the original manuscript, not only from her incredible attention to the fine points of the English language, but also because her expert knowledge of quantum mechanics more than once pointed me toward a clearer way to frame some technical aspect of physics.

Finally, I must thank my family – my parents, Harry and Mary, for providing as great an environment in which to come of age as I can imagine, my children, John, Thomas, and Maddy, who are a constant source of joy and surprise, and especially my wife, Kris, steadfast companion and best friend whose support and encouragement throughout these past 36 years have made my mathematical career possible.

Tallahassee, Florida

Prolegomenon

> Never in the history of science has there been a theory which has had such a profound impact on human thinking as quantum mechanics; nor has there been a theory which scored such spectacular successes in the prediction of such an enormous variety of phenomena (atomic physics, solid state physics, chemistry, etc.). Furthermore, for all that is known today, quantum mechanics is the only consistent theory of elementary processes.
>
> Thus although quantum mechanics calls for a drastic revision of the very foundations of traditional physics and epistemology, its mathematical apparatus or, more generally, its abstract *formalism* seems to be firmly established. In fact, no other formalism of a radically different structure has ever been generally accepted as an alternative. The *interpretation* of this formalism, however, is today, almost half a century after the advent of the theory, still an issue of unprecedented dissension. In fact, it is by far the most controversial problem of current research in the foundations of physics and divides the community of physicists and philosophers of science into numerous opposing "schools of thought."
>
> Max Jammer
> in the Preface to *The Philosophy of Quantum Mechanics*
> (1974)

A Wealth of Mathematics

One of the unexpected windfalls for the mathematician who studies quantum mechanics is that here, in this one topic, the umbrella theory of the whole of physics, one finds a host of modern, sophisticated mathematics that is pertinent to its sound telling. This serves to clothe the bones of the theoretical mathematics that the student learns in his or her graduate studies with the concrete flesh of the physical world. Here the abstraction and generality of graduate pure mathematics take on a specificity of meaning in its application to the understanding of the workings of the micro-world. This helps to solidify in the student the effectiveness of his or her craft and serves to showcase the interconnectedness of many topics that have been presented in isolation from one another. There is no other theory within the natural sciences that surveys in its sound telling so much of the landscape of modern pure mathematics. Here I present a brief listing of the disciplines and subdisciplines within both classical and modern mathematics that make an appearance in a mathematically honest study of quantum mechanics. All three primary divisions of pure mathematics – analysis, algebra, and geometry/topology – are suitably represented in this listing. The student of quantum mechanics cannot be expected to have mastered all the topics in the list, not even the majority of them, but he or she should be willing to work through just enough to apply to the discussion at hand.

Analysis

Of the three broad divisions, perhaps analysis has the greatest claim on quantum mechanics. The very playing field of Schrödinger's wave mechanics is the Hilbert space of square-integrable Lebesgue measurable functions. In the axiomatic treatment of the theory, other Hilbert spaces make their appearance as important instantiations of the axioms. It is not surprising that the isometry of $L^2(\mathbb{R}^n)$, known as the Fourier transform, makes a central appearance as the transformation from the position to the momentum representation of the theory. As much of the language of the elementary applications of quantum mechanics is expressed in the form of partial differential equations, there is an abundance of classical special function theory in the mix – the polynomials of Hermite, Laguerre, and Legendre and their associated functions; spherical harmonics, general hypergeometric functions, and the gamma function. Contour integration makes its appearance as do stereographic projection, Green's formula, the general Stokes theorem, Gaussian integrals, and the differential operators of Laplace, d'Alembert, and Dirac.

It is not an overstatement to say that functional analysis is at the very core of a rigorous treatment of quantum mechanics. Self-adjoint operators in all their subtlety are pivotal for the development of the theory of measurement, and unitary operators for a description of quantum evolution and for a clarification of the role of symmetry in determining conservation laws in the theory. Not surprisingly, the two great theorems of the subject, the spectral theorem for self-adjoint operators and Stone's theorem on generators for one-parameter unitary groups, are crucial for providing rigor to quantum measurement, unitary evolution, and quantum conservation laws. Distribution theory and generalized functions, with the machinery of Schwartz spaces and test functions and tempered distributions, appear both in the development of the theory of the Fourier transform and as a way to explain the *lingua franca* of the subject, the Dirac calculus with its bra-ket notation for state vectors.

Finally, the calculus of variations and the Euler equation are important in the development of the Lagrangian approach, and canonical transformations for the introduction of the Hamiltonian approach, and Noether's beautiful theorem is important for understanding how conservation laws arise from symmetries.

Algebra

Groups, vector spaces, and algebras all have their place in informing and building quantum theory. Infinite-dimensional vector spaces are really the stuff of functional analysis, but their finite-dimensional cousins show up as state spaces of finite-dimensional quantum systems and as representation spaces of groups and algebras. Lie groups, Lie algebras, and their real, complex, and quaternionic representations are used to develop a proper theory of spin and angular momentum, and unitary representations appear in discussions of symmetry and conservation laws. Of primary importance are the-lower dimensional classical matrix groups and algebras, particularly $\mathrm{SU}(2)$, $\mathrm{SO}(3)$, $\mathrm{SO}(4)$, $\mathrm{Sp}(1)$, $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$, the quaternions $\mathbb{H}$, the Lie algebra $\mathfrak{su}(2)$, the matrix algebras $\mathbf{M}_2(\mathbb{K})$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$, and $\mathbf{M}_4(\mathbb{C})$. Clifford algebras as well as spin groups and spin representations play important roles in the Pauli and Dirac theories of fermions. Group characters are used in solving the momentum space equation for the hydrogenic potential. Representations of the Lorentz and Poincaré groups uncover some intricacies of special relativity, and tensor products are used to construct state spaces for multi-part quantum systems and as a tool for incorporating novel aspects into a known quantum system as, for instance, in the Pauli theory of spin. Of particular interest are the exterior and symmetric graded subalgebras of the tensor algebra of a Hilbert space and their completions, used as multi-particle quantum state spaces for fermions and bosons.

Geometry/Topology

Differentiable and Riemannian manifolds make their appearance along with some of their associated machinery – tangent and cotangent bundles, tensor fields and differential forms, covariant derivatives and connections. These are used in placing some traditional topics of physics into a global, modern setting. This occurs in discussions of Lagrangian and Hamiltonian mechanics, of gauge symmetry of the electromagnetic field, and, to a lesser extent, in some comments on special relativity. Covering spaces arise in the context of examining the topology of SU(2) and SO(3), as well as in the completely unrelated topics of the Aharonov–Bohm effect and Dirac monopoles. The quantum version of electromagnetic theory may be placed in the sophisticated setting of complex line bundles and Koszul connections, where all the gauge equivalent theories are seen as trivializations of a single theory. This latter version using complex line bundles is rather esoteric, but the payoff in mastering it is the unity this sophisticated approach brings to the theory.

Starred Sections

Lest this listing of modern mathematics should frighten the reader, many of these topics, particularly the ones from topology and geometry, do not appear in this book. They are needed for building the mathematical edifice of relativistic quantum mechanics, which is avoided here. Some lectures and sections in this book that employ more advanced topics are marked with a star, $\star$, and may be skipped without affecting the flow of the book.

The Lie of the Land

This is a large book with an abundance of topics. I think it may be as well to present something of a guided tour for the prospective reader. Before that, though, allow a bit of advice from an old hand at learning mathematics and physics. The strict training of many students of mathematics imprints a certain psychological pressure on their practice of learning a new topic. Since the rallying cry of pure mathematics, especially in the foundational courses at the advanced undergraduate and beginning graduate levels, is rigorous argument – nothing accepted without airtight proof argued from axioms or theorems – the student often enters the second year of graduate work with a psychological need to learn any new technical subject linearly, accepting nothing until its proof is understood. Now that the student is conversant with the craft of constructing, reading, and understanding proofs, this psychological need for linear learning and complete rigor should be broken. Most research in mathematics takes place at a frontier so far removed from the foundational courses that one cannot afford to dot every i and cross every t in order to get to the frontier. The student must learn to understand the statements of well-established results and how to use them, hoping to place them in his or her mathematical toolbox, and use the fact that these tools are very useful to motivate him or her to go back later, when time affords the luxury, to dot the i's and cross the t's. Learning how a result can be used to derive new results and to bring understanding and organization to a technical subject, even without understanding the rigors of the proof of the result, can be a great motivator for delving deeply into that topic later. So, my advice to the reader is to allow himself or herself an indulgence, namely, to browse this book and pick and choose parts that interest him or her. Read a chapter without necessarily having read the background material needed for that chapter; skip the more technical discussions on some of the minutiae in order to get to the heart of the physical theory; do not read this book linearly! I think you will learn more and have a more satisfying experience with the book if you heed this advice. With this in mind, I begin the guided tour.

A Guide to this Book

The book you are holding comprises 32 chapters, which from now on are called lectures. It presents the basic results of the nonrelativistic theory of quantum mechanics. While all the important theoretical features of quantum mechanics make their appearance, including some nontraditional topics, the treatment largely neglects some issues found in standard texts. For example, I make little mention of the experimental work that motivated the development of the theory. I omit the one-dimensional scattering discussion of wave packets hitting potential barriers and the several standard perturbation techniques used to approximate solutions to the Schrödinger equation. And, finally, I largely avoid some standard metaphysical musings on the subject that I find opaque.

On this last point, I do not mean that philosophical and interpretative issues are abandoned altogether. In fact, I have a healthy respect for these issues and encourage the student to explore them here and on his or her own. I only mean that some claims found in some of the standard texts that champion the standard (Copenhagen) interpretation of the theory are either, on the one hand, rather abstruse or unintelligible, or, on the other hand, outright dogmatic and lend nothing to an understanding of the theory.

Lecture 1 presents a comparison of the classical treatment of a physical system by Newtonian mechanics and the mathematical treatment using the rules of quantum mechanics. This is designed to illustrate the striking differences between the two approaches, both in methodology and in results. The system chosen for presentation is that of a particle in a central quadratic potential – the simple harmonic oscillator – whose classical analysis is familiar from elementary mathematics courses. The quantum treatment should seem rather odd and its meaning will not be apparent from the mathematical analysis applied. It will not even be clear from the quantum analysis that the results say anything about the motion of the particle in this central potential. This state of affairs is remedied by interpreting the quantum results using an understanding that developed along with the theory and imprints meaning on the quantum calculations in terms of measurement probabilities. There follows a more philosophical discussion on foundational issues that lays out for the reader some of the very interesting and deep questions that arise concerning what quantum mechanics says about the nature of reality at the micro-level. My hope is that this initial lecture will give that reader who is a novice to the subject an intellectual jolt that will inspire him or her to delve deeply into the mathematical and philosophical issues of this intriguing theory.

The book's treatment of quantum mechanics proper begins with *Lecture 2*. Here the basic mathematical structure of the theory is encased in a set of axioms that are rather minimal with respect to assumptions. Over the next three lectures these axioms are unpacked and expanded, and the mathematics necessary to bring rigor to the subject is explained. This entails an excursion into functional analysis with emphases on self-adjoint operators and the subtleties of their definition, the spectral theory of these operators, and rigged Hilbert spaces. As the book moves through these mathematical issues, it concurrently develops the physical interpretation necessary to give meaning to the quantum calculations. In particular, the probability interpretation of the state vector is presented, quantum measurement is discussed along with the knotty problem this leaves unresolved, and the Dirac bra-ket calculus is presented and placed on a sound mathematical footing. This takes us through *Lecture 5*.

Lecture 6 presents a review of classical mechanics, including the Lagrangian and Hamiltonian formulations, as well as Noether's theorem on conserved quantities. For the student who is unfamiliar with these topics, this lecture should serve as an accessible introduction with enough detail sufficient for their use in the development of quantum mechanics. *Lecture 7* on Jacobi's contributions to classical mechanics is a bonus lecture that completes the classical picture but may be skipped without affecting the flow of the book. *Lecture 8* relates the quantum development to the classical, suggesting how the classical treatment of the macro-world of everyday objects emerges from the quantum picture.

Schrödinger's wave mechanics is introduced briefly in Lecture 2 as the most important instantiation of the minimal axioms for quantum mechanics, but is developed rather sparsely over the following few lectures. It is developed more fully in *Lectures 9, 10, 11, 15, 16, 17,* and *19*. *Lecture 9* attempts to clear away a bit of the fog that has accumulated over popular accounts of the Heisenberg uncertainty principle, and it introduces minimal uncertainty states as well as the Fourier transform. The Fourier transform is reviewed quite carefully in *Lecture 10* and used to introduce the momentum representation of wave mechanics. The first fully quantum mechanical calculation is performed in *Lecture 11*, giving details of the quantum analysis of the simple harmonic oscillator reported in the first lecture. A thorough treatment of the hydrogenic potential, begun in *Lecture 14*, is presented in *Lectures 15, 16,* and *17*. In particular, the usual quantum mechanical treatment appears in *Lectures 14* and *15*; an unusual treatment is presented in *Lecture 16*, where the hidden symmetry of the hydrogenic potential is revealed from an analysis of the momentum space equation. This is followed with two solutions to the momentum space equation in *Lecture 17*, the first derived from Fock's original treatment of the equation and the second from a modern treatment based on group characters. Finally, Pauli's ad hoc addition to the quantum formalism to include spin, his *spinor theory*, is introduced in *Lecture 19* as an example of how to use the formalism of tensor products to include novel phenomena in a quantum theory.

Angular momentum operators are treated in *Lectures 12, 13, 14,* and *18*, with bonus material in *Lecture 20*. The orbital angular momentum operators are defined in *Lecture 12* by quantizing the classical angular momentum formulae of classical mechanics, after which the eigenvalues and eigenvectors of these operators are determined. Realizing that the representation theory of the Lie algebra $\mathfrak{su}(2)$ is central to a mathematical understanding of these operators, *Lecture 13* introduces the Pauli spin matrices in the context of the algebra $\mathfrak{su}(2)$ and reinterprets the results of the preceding lecture in light of representation theory. This lecture becomes a short course on the low-dimensional Lie groups and their Lie algebras, which have much currency in modern physics. In particular, detailed developments of the quaternion algebra $\mathbb{H}$, the 3-sphere group $\mathbf{S}^3$, and the Lie groups SU(2), SO(3), and SO(4), as well as their corresponding Lie algebras, are presented. The representation theory of SU(2) and SO(3) is worked out in detail and the topological structure of these groups is uncovered. This mathematical work pays off in *Lecture 14*, where the central force problem is analyzed. Here the representation theory of SO(3) is found to be crucial for understanding the finer points of the space of spherical harmonics, which themselves arise in the analysis of the central force problem. This material is necessary for a complete understanding of the hydrogenic potential analyzed in the lecture following. After a foray into the analysis of the hydrogenic potential, *Lecture 18* returns to the subject of angular momentum operators.

Up to this point the full integral representations of $\mathfrak{su}(2)$ have been used to build a quantum theory of orbital angular momentum, but this begs the question of whether the half-integral representations have anything to add to the quantum mix. After presenting the addition rules for angular momentum operators, *Lecture 18* continues by analyzing the simplest half-integral representation, the spin one-half system, that turns out to be crucial for an understanding of spin in quantum theory. Spin is a purely quantum mechanical property of particles that has no classical counterpart, and is integrated into the quantum development in the next lecture by incorporating the spin one-half representations via Pauli's spinor theory. With this introduction of spin into the quantum formalism, it seems natural to pause for a moment and include a mathematical lecture, *Lecture 20*, on Clifford algebras and spin representations. This material is not needed for the remainder of the development and may be skipped, but I have decided to place it here to position the Pauli theory in a context that is modern and one that anticipates the more natural Dirac theory, a critical ingredient of relativistic quantum mechanics.

Thus far in the development of the theory I will have dealt only with quantum systems that are indivisible or systems with a single quantum particle. *Lectures 21* through *26* explain how to incorporate multi-part