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SYMMETRY IN GRAPHS

This is the first full-length book on the major theme of symmetry in graphs. Forming part of algebraic graph theory, this fast-growing field is concerned with the study of highly symmetric graphs, particularly vertex-transitive graphs, and other combinatorial structures, primarily by group-theoretic techniques. In practice, the street goes both ways and these investigations shed new light on permutation groups and related algebraic structures.

The book assumes a first course in graph theory and group theory but no specialized knowledge of the theory of permutation groups or vertex-transitive graphs. It begins with the basic material before introducing the field's major problems and most active research themes in order to motivate the detailed discussion of individual topics that follows. Featuring many examples and with over 450 exercises, it is an essential introduction to the field for graduate students and a valuable addition to any algebraic graph theorist's bookshelf.

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Preface

Symmetry and its absence, **asymmetry**, is one of the core concepts in the arts, science, or daily life, for that matter. In everyday language, symmetry – from the Greek "$\sigma\upsilon\mu\mu\epsilon\tau\rho\iota\alpha$ = symmetria," agreement in due proportion, arrangement – usually refers to a sense of beauty, of proportional harmony, and of balance. In trying to present the concept of symmetry in as general a setting as possible, one can think of it as a measure of the inner structural stability of the object in question when confronted with possible external intrusions. When the latter exceed the acceptable robustness threshold and thus break the innate symmetry, a transformation of the object occurs. Given a new life of enriched complexity, the object finds itself in an environmental absence of symmetry and starts a slow but steady ascent to a balanced state of a brand-new stability.

In mathematics, the notion of symmetry, although present in many different areas, is mostly studied by means of **groups** and is therefore arguably an inherently algebraic concept – namely, the set of all transformations/symmetries of a given mathematical object that preserves its inner structure forms a group. Knowing the full set of symmetries of an object is important because it provides the most complete description of its structure.

Historically, many of the most fruitful techniques for studying groups have been developed within the framework of permutation groups and, more generally, **group actions**, preferably on sets enjoying additional inner structure. Among these, transitive actions are perhaps the most natural ones for they represent the building blocks for actions in general. This interplay is nicely recaptured in the combinatorial setting of relational structures, most notably in **graphs**, as the simplest manifestation of such structures. Being one of the core concepts essential both to understanding natural phenomena and the dynamics of social systems and to providing a theoretical framework for efficient mass communication, relational structures (and graphs in particular) with high levels of symmetry are sought for their optimal behavior and high performance.

A mathematical model capturing this situation, and a common thread of this book, is the concept of a **vertex-transitive graph** (together with its counterparts, such as edge-transitive graphs and arc-transitive graphs), and the underlying mathematical discipline is referred to as **algebraic graph theory**. The benefits are twofold. As much as the group-theoretic properties are reflected in the associated vertex-transitive graphs, the various structural properties of the latter are best studied using group-theoretic tools: This is precisely the viewpoint that this book is meant to uphold. Its content is a virtual two-way traffic on a one-way street: While pursuing symmetry properties of combinatorial objects is the main objective of this book, we will, as a byproduct, obtain additional information and understanding of the accompanying permutation groups and other algebraic objects involved in the process.

This philosophy is best reflected in a detailed motivational analysis of the various actions of the alternating group A_5 on the Petersen graph – each corresponding to a specific subgroup of A_5, with the natural action on its vertex set being just one of them. The analysis, which identifies with each subgroup of A_5 a corresponding combinatorial substructure of the Petersen graph, is supposed to serve as an introductory argument. A precursor to the central theme of this book is that mutual, algebraic, and combinatorial benefits are obtained by addressing the intrinsically algebraic concept of symmetry in a combinatorial setting. This analysis is the content of Sections 2.5 and 2.6, and, as it needs only very basic group-theoretic and graph-theoretic concepts, the reader familiar with basic terminology is welcome to jump right into it and start there.

This book has been written assuming that the reader has had a first course in group theory including the Sylow Theorems, as well as a first course in graph theory. A first course in permutation group theory, the main algebraic tools in this text, is **not** assumed. We will develop the permutation group theory tools that we need as we go, with proofs of some (usually longer) results omitted (for example, the O'Nan–Scott Theorem). Basic knowledge of fields and finite fields is assumed (for example, the definition of a field, and the knowledge that finite fields exist and that the group of units in a finite field is cyclic) in some parts as well as some basic linear algebra. Some knowledge of basic number theory would be useful (but mostly what is used in number theory is simply modular arithmetic as seen in group theory).

The book is organized as follows. The first five chapters are on material that we believe everyone who works in this area of algebraic graph theory should have at least some familiarity with. So our intention is that these chapters should more or less be covered in order. That is, each successive chapter assumes you are familiar with the preceding chapters.

Chapter 6 covers additional families of graphs that are fairly common but perhaps not as common as the families in the first five chapters. It is designed to be independent of the material that follows it, and each section of Chapter 6 is independent of the other sections in Chapter 6, so the reader could skip this chapter and only come back to it if needed. Some chapters after Chapter 6 do make use of some material in some sections of Chapter 6.

Chapters 7–13 are each on a different problem in the area of algebraic graph theory. For practical reasons, Chapter 7 should probably be read before Chapter 8. Chapter 7 is on the isomorphism problem for Cayley digraphs and Chapter 8 is on automorphism groups of (mainly) Cayley digraphs. In practice, the isomorphism problem for a fixed group G has been solved before the corresponding automorphism group problem, as the isomorphism problem can be solved using properties of the automorphism group.

Chapter 9 discusses some classification results for vertex-transitive graphs, focusing on classifications of certain orders. Section 9.3 uses results from Section 7.6. The remaining chapters can be read more or less independently. Chapters 10, 11, and 12 focus on topics already encountered in Chapter 3, namely, symmetric graphs, the Hamiltonian problem, and the semiregular elements in automorphism groups of vertex-transitive digraphs. In Chapter 13 we consider graphs with other kinds of symmetry not yet encountered. Finally, in Chapter 14 we discuss some problems we believe deserve a closer look in the future.

As permutation group theory is not a prerequisite for this book, it contains enough permutation group theory to form a short course in that subject. However, as our focus is on graphs with symmetry, the permutation group theory is usually given as needed for a particular graph-theoretic topic. So the permutation group theory is not provided in a self-contained part of the book.

Thanks are owed to many people who have actively contributed to the development of this book. Manuscript forms were used as the basis of several courses taught at the University of Primorska over the last several years. Additionally, quite a few mathematicians have read various parts of the manuscript and offered suggestions on improvements. For this, our thanks go to Rachel Barber, Ademir Hujdurović, István Kovács, Klavdija Kutnar, Štefko Miklavić, Luke Morgan, Rok Požar, Primož Šparl, and Ágnes Szalai.