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Multivariate Approximation

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Preface

The twentieth century was a period of transition from univariate problems (i.e., single-variable problems) to multivariate problems in a number of areas of mathematics. In many cases this step brought not only new phenomena but also required new techniques. In some cases even the formulation of a multivariate problem requires a nontrivial modification of a univariate problem. For instance, the problem of the convergence of the multivariate trigonometric series immediately encounters the question of which partial sums we should consider; there is no natural ordering in the multivariate case. In other words: what is a natural multivariate analog of the trigonometric polynomials? In answering this question mathematicians have studied different generalizations of the univariate trigonometric polynomials: those with frequencies from a ball, a cube, a hyperbolic cross.

Multivariate problems turn out to be much more difficult than their univariate counterparts. The main goal of this book is to demonstrate the evolution of theoretical techniques from the univariate case to the multivariate case. We do this using the example of the approximation of periodic functions. It is justified historically and also it allows us to present the ideas in a concise and clear way. In many cases these ideas can be successfully used in the nonperiodic case as well.

We concentrate on a discussion of some theoretical problems which are important in numerical computation. The fundamental problem of approximation theory is to resolve a possibly complicated function, called the target function, into simpler, easier to compute, functions called approximants. Generally, increasing the resolution of the target function can be achieved only by increasing the complexity of the approximants. The understanding of this tradeoff between resolution and complexity is the main goal of approximation theory. Thus the goals of approximation theory and numerical computation are similar, even though approximation theory is less concerned with computational issues. Approximation and computation are intertwined and it is impossible to understand fully the possibilities in numerical computation without a good understanding of the elements of approximation theory. In particular, good approximation methods (algorithms) from approximation theory find applications in image processing, statistical estimation, regularity for PDEs, and other areas of computational mathematics.

We now give a brief historical overview of the challenges and open problems in approximation theory, with emphasis on multivariate approximation. It was understood at the beginning of the twentieth century that the smoothness properties of a univariate function determine the rate of approximation of this function by polynomials (trigonometric in the periodic case and algebraic in the nonperiodic case). A fundamental question is: what is a natural multivariate analog of the univariate smoothness classes? Different function classes have been considered in the multivariate case: isotropic and anisotropic Sobolev and Besov classes, classes of functions with bounded mixed derivative, and others. The next fundamental question is: how do we approximate functions from these classes? Kolmogorov introduced the concept of the n-width of a function class. This concept is very useful in answering the above question. The Kolmogorov n-width is a solution to an optimization problem where we optimize over n-dimensional linear subspaces. This concept allows us to understand which n-dimensional linear subspace is the best for approximating a given class of functions. The rates of decay of the Kolmogorov n-width are known for the univariate smoothness classes; in some cases, exact values are known. The problem of the rates of decay of the Kolmogorov n-width for the classes of multivariate functions with bounded mixed derivatives is still an open problem. We note that the function classes with bounded mixed derivatives are not only an interesting and challenging object for approximation theory but are also important in numerical computations. M. Griebel and his group have used approximation methods designed for these classes in elliptic variational problems. Recent work of H. Yserentant on new regularity models for the Schrödinger equation shows that the eigenfunctions of the electronic Schrödinger operator have a certain mixed smoothness similar to that of the bounded mixed derivative. This makes approximation techniques developed for classes of functions with bounded mixed derivatives a proper choice for a numerical treatment of the Schrödinger equation.

Following the Kolmogorov idea of the n-width the optimization approach to approximation problems has been accepted as a fundamental principle. We follow this fundamental principle in this book. We will demonstrate the development of ideas and techniques when we pass from the univariate case to the multivariate case, and we will solve some optimization problems for the multivariate approximation. We find the correct order of decay for a number of asymptotic characteristics of smoothness function classes. We provide the rate of decay, in the sense of order, as a function of the complexity of the approximation method. For example, in the case of approximation by elements of a linear subspace the complexity parameter is the dimension of the subspace. In the case of numerical integration the complexity parameter is the number of points (knots) of the cubature formula. Solving the problem of the correct (in the sense of order) decay of an asymptotic characteristic, we need to solve two problems: (U) prove the upper bounds and (L) prove the lower bounds. Very often the technique used to solve the problem (U) is very different from the one for solving (L). We present both the classical, well-known, techniques and comparatively recent modern techniques. Usually, when we solve an approximation problem, we use a combination of fundamental results (theories) and special tricks. In this book the fundamental results from harmonic analysis, for instance, the Littlewood–Paley theorem, the Marcinkiewicz multiplier theorem, the Hardy–Littlewood inequality, the Riesz–Thorin theorem, and the Hausdorff–Young theorem are widely used. It turns out that in addition to these classical results we need new fundamental results in order to solve the multivariate approximation problems for mixed smoothness classes. We develop and demonstrate this new technique here in detail. Let us mention three examples of this new fundamental technique: (I) an embedding-type inequality proved in §3.3.3 (see Theorem 3.3.6); (II) the volume estimates of sets of coefficients of trigonometric polynomials (see §3.2.6 and Chapter 7); (III) the greedy approximation (see Chapters 8 and 9).

Motivated by practical applications we study the following theoretical problem: how do we replace in an optimal way an infinite-dimensional object (a function class) by a finite or finite-dimensional object? The theory of widths, in particular, the Kolmogorov widths, addresses the problem of approximation from a finite-dimensional subspace. In addition to the Kolmogorov width we employ the linear width and the orthowidth (Fourier width). In this study we use techniques (I) and (II). Technique (II) has proved to be very useful in the proof of the lower bounds. There are still open problems in finding the correct orders of decay of the above widths in case of the mixed smoothness classes.

Discretization is a very important step in making a continuous problem computationally feasible. The problem of the construction of satisfactory sets of points in a multidimensional domain is a fundamental problem of mathematics and, in particular, computational mathematics. We note that the problem of arranging points in a multidimensional domain is also a fundamental problem in coding theory. It is a problem on optimal spherical codes. This problem is equivalent to the problem from compressed sensing on building large incoherent dictionaries in $\mathbb{R}^d$. A very interesting and difficult problem is to provide an explicit (deterministic) construction of a large system with small coherence. The optimal rate in this problem is still unknown (see Temlyakov, 2011, Chapter 5, for further discussion).

A prominent example of a discretization problem, discussed in detail in this book, is the problem of numerical integration. It turns out that, contrary to numerical integration in the univariate case (see §2.4) and in the multivariate case of anisotropic smoothness classes (see §3.6), where regular grid methods are optimal (in the sense of order), in the case of the numerical integration of functions with mixed smoothness regular grid methods are very far from being optimal. Numerical integration in mixed smoothness classes requires deep number-theoretical results for constructing optimal (in the sense of order) cubature formulas (see Chapter 6). In addition to number-theoretical methods, technique III is also of use here.

Another example of a classical discretization problem is the problem of metric entropy (covering numbers and entropy numbers). Bounds for the ε-entropy of function classes are important in themselves and also have important connections to other fundamental problems. For instance, the problem of the ε-entropy of some classes of functions with bounded mixed derivatives is equivalent to the fundamental small ball problem from probability theory. This problem is still unresolved in dimensions greater than two (see Temlyakov, 2011, Chapter 3, and Dinh Dung *et al.*, 2016 for further discussion). We obtain the correct orders of decay of the entropy numbers of mixed smoothness classes in Chapter 7. Technique II plays a fundamental role in proving the lower bounds.

The above discussion demonstrates that multivariate approximation theory in a classical setting has close connections with other areas of mathematics and has many applications in numerical computation. Recently, driven by applications in engineering, biology, medicine, and other areas of science, new and challenging problems have appeared. The common feature of these problems is very high dimensions. Classical methods developed in multivariate approximation theory may work for moderate dimensions, say, up to 40 dimensions. Many contemporary numerical problems, however, have dimensions which are really large – sometimes in the millions. Classical methods do not work for such enormous dimensions. This is a rapidly developing and hot area of mathematics and numerical analysis, where researchers are trying to understand which new approaches may work. A promising contemporary approach is based on the concept of sparsity and nonlinear m-term approximation. We present the corresponding results in Chapters 8 and 9 of this book.

The fundamental question of nonlinear approximation is how to devise effective constructive methods (algorithms) of nonlinear approximation. This problem has two levels of nonlinearity. The first is m-term approximation with regard to bases. In this problem one can use the unique function expansion with respect to a given basis to build an approximant. Nonlinearity enters by looking for m-term approximants with terms (i.e. basis elements in the approximant) that are allowed to depend on a given function. Since the elements of the basis used in the m-term approximation are thus allowed to depend on the function being approximated, this type of approximation is very efficient. On the second level of nonlinearity, we replace a basis by a more general system, which is not necessarily minimal (for example, a redundant system, or a dictionary). This setting is much more complicated than the first (the bases case); however, there is a solid justification due to the importance of redundant systems in both theoretical questions and in practical applications. Technique III turns out to be very useful for approximation at both levels of nonlinearity. In this book we are primarily interested in the trigonometric approximation. A very interesting phenomenon was observed recently. It turns out that nonlinear algorithms, in particular the Chebyshev greedy algorithms, designed for approximation with respect to redundant systems, work better than algorithms, in particular the thresholding greedy algorithm, designed for bases when they are applied to a trigonometric system. We discuss this phenomenon in detail in Chapter 8.

Above we discussed a strategy based on the optimization principle, which we will apply for finding optimal (in the sense of order) finite dimensional subspaces (theory of the widths) and optimal (in the sense of order) discretization (numerical integration, entropy). In addition to the optimization principle we study another fundamental principle – universality. In Chapters 5 and 6 we illustrate the following general observation. Methods of approximation and numerical integration which are optimal in the sense of order for classes with mixed smoothness are universally applicable for the collection of anisotropic smoothness classes. This gives an a posteriori justification for the thorough study of classes of functions with mixed smoothness. The phenomenon of saturation is well known in approximation theory (DeVore and Lorentz, 1993, Chapter 11). The classical example of a saturated method is the Fejér operator for approximation of the univariate periodic functions. In the case of a sequence of Fejér operators, saturation means that the approximation order by Fejér operators of order n does not improve over the rate $1/n$ even if we increase the smoothness of the functions under approximation. Methods (algorithms) that do not have the saturation property are called unsaturated. The reader can find a detailed discussion of unsaturated algorithms in approximation theory and in numerical analysis in the survey paper Babenko (1985). We point out that the concept of smoothness becomes more complicated in the multivariate case than it is in the univariate case. In the multivariate case a function may have different smoothness properties in different coordinate directions. In other words, functions may belong to different anisotropic smoothness classes (see Chapter 3). It is known (see Chapter 3) that the approximation characteristics of anisotropic smoothness classes depend on the average smoothness and that optimal approximation methods depend on the anisotropy of classes. This motivated a study, in Temlyakov (1988c) of the existence of an approximation method that works well for all anisotropic smoothness classes. The problem is that of the existence of a universal method of approximation. We note that the universality concept in learning theory is very important and is close to the concepts of adaptation and distribution-free estimation in nonparametric statistics (Györfy *et al.*, 2002, Binev *et al.*, 2005, Temlyakov, 2006). We discuss universality in approximation theory in §5.4 and universality in numerical integration in §6.8.

We now give a brief description of the book by chapters.

Chapter 1. Approximation of Univariate Functions This chapter contains classical results of approximation theory: the properties of trigonometric polynomials and approximation by trigonometric polynomials. The selection of the material for this chapter was dictated by further developments and applications in the multivariate approximation.

Chapter 2. Optimality and Other Properties of the Trigonometric System This chapter contains classical results on the Kolmogorov, linear, and Fourier widths of the univariate smoothness classes. Discretization techniques and the fundamental finite-dimensional results are discussed in detail in this chapter. Also, classical results on the convergence of Fourier series are presented here. The book by DeVore and Lorentz (1993) contains a comprehensive presentation of the univariate approximation. The presentation in Chapters 1 and 2 is aimed towards multivariate generalizations. It is somewhat close to the presentation in Temlyakov (1993b).

Chapter 3. Approximation of Functions from Anisotropic Sobolev and Nikol'skii Classes This chapter is the first step from the univariate approximation to the multivariate approximation. The approximation technique discussed here is mostly similar to the univariate technique. Results of this type are typically presented in books, such as Nikol'skii (1969), on function spaces. However, we include in this chapter some nontrivial embedding-type inequalities and estimates for the volumes of sets of Fourier coefficients of the multivariate trigonometric polynomials, which are frequently used in further chapters.

Chapter 4. Hyperbolic Cross Approximation This is one of the main chapters on the linear approximation theory of functions with mixed smoothness. In the sense of its settings it is parallel to Chapter 1. This parallelism in settings allows us to demonstrate a deep difference in technique between univariate polynomial approximation and hyperbolic cross polynomial approximation.

Chapter 5. The Widths of Classes of Functions with Mixed Smoothness This is the other main chapter on the linear approximation theory of functions with mixed smoothness. The relation between this chapter and Chapter 2 is the same as that between Chapters 4 and 1.

Chapter 6. Numerical Integration and Approximate Recovery This is the third main chapter on the linear approximation theory of functions with mixed smoothness. This chapter is important from the point of view of applications. Also, the technique for numerical integration developed in this chapter is very different from that for univariate numerical integration. It is based on deep number-theoretical constructions and on the general theory of greedy algorithms. The numerical integration of classes of functions with mixed smoothness has attracted a lot of attention recently. There are many books on discrepancy theory that are related to this chapter. However, the development in terms of numerical integration is more general than a discrepancy-type presentation. Roughly speaking, discrepancy theory corresponds to the case of smoothness equal to 1 and equal weights in cubature formulas, while numerical integration theory considers the whole range of smoothness and general cubature formulas.

Chapter 7. Entropy This is the first chapter devoted to nonlinear approximation theory. We include here classical results on the entropy numbers of finite-dimensional compacts. The main new ingredient of this chapter is a study of the entropy numbers of classes of functions with mixed smoothness.

Chapter 8. Greedy Approximation This chapter contains a brief introduction to greedy approximation in Banach spaces and a recent result on the Lebesgue-type inequality for the Chebyshev greedy algorithm (CGA) with respect to special dictionaries. In particular, this result implies that the CGA provides almost ideal (up to a $\log m$ factor) m-term trigonometric approximation for all functions. Our introduction to greedy approximation in Banach spaces follows the lines of Temlyakov (2011). The Lebesgue-type inequality is also a recent result (see Temlyakov, 2014).

Chapter 9. Sparse Approximation This is one of the most important chapters of the book. Our main interest in this chapter is to study sparse approximation problems for classes of functions with mixed smoothness. We discuss in detail m-term approximation with respect to the trigonometric system. We use techniques based on a combination of results from the hyperbolic cross approximation, which were obtained in the 1980s and 1990s (and are presented in Chapters 3–5 and 7), and recent results on greedy approximation (given in Chapter 8) to obtain sharp estimates for the best m-term approximation.

Appendix. The Appendix contains classical inequalities and results from harmonic analysis that are often used in this text.

The book is devoted to the linear and nonlinear approximation of functions with mixed smoothness. Both Temlyakov (1986c) and Temlyakov (1993b) contain results on the linear approximation theory of such classes. At present there are no books on the nonlinear approximation theory of these classes. In addition to the results treated in Temlyakov (1993b), we describe in this book substantial new developments in the linear approximation theory of classes with mixed smoothness. This makes the book the most complete text on the linear approximation theory of these classes. Further, it is the first book on the nonlinear approximation theory of such classes. The background material included in Chapters 1–3 makes the book self-contained and accessible for readers with graduate or even undergraduate level mathematical education. The theory of the approximation of these classes and related questions are important and actively developing areas of research. There are still many unresolved fundamental problems in the theory. Many open problems are formulated in the book.

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