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Introduction

As already mentioned above a main tool to identify and investigate *solvable* dynamical systems—including N-body models describing the evolution in the *complex* plane (or equivalently in the *real* Cartesian plane) of N point particles interacting *nonlinearly* among themselves and moving according to *Newtonian* equations of motion ("accelerations equal forces")—are the relations among the *time-evolutions* of the N *zeros* and those of the N *coefficients* of a monic *time-dependent* polynomial of (*arbitrary*) degree N. This approach was introduced decades ago [16] and has been extensively used since—see, for instance, the two books [20] and [27] and references therein—but the technique to implement it was essentially restricted to the consideration of the time evolution of the *zeros* of a (monic) polynomial the *coefficients* of which evolve according to a system of *linear* Ordinary Differential Equations (ODEs). The elimination of this restriction—to *linear* evolutions of the *coefficients*—was recently made possible by the key formulas reported in Chapter 2, opening thereby the way to the identification of many more *solvable/integrable* N-body problems, as described in Chapter 4 and demonstrated by the examples reported there. This approach has also been extended to the identification and investigation of *solvable nonlinear Partial Differential Equations* (PDEs: see Chapter 5) and to equations of evolution in *discrete time* (see Chapter 7).

Moreover—as described in Chapter 6—this development naturally led to the idea of *generations* of monic polynomials of degree N—with the N *coefficients* of the polynomials of the *next* generation identified with the N *zeros* of the polynomials of the *current* generation—and to the related possibility to identify *endless hierarchies* of *nonlinear solvable* dynamical systems, including, again, N-body problems describing the evolution in the *complex* plane (or, equivalently, in the *real* Cartesian plane) of N point-particles interacting *nonlinearly* among themselves and moving according to *Newtonian* equations of motion ("accelerations equal forces").

Some other developments that have also followed from the key formulas reported in Chapter 2 are treated in other chapters and sections of this book: the interested reader may immediately identify them from the titles of the relevant chapters, sections and subsections, as displayed in the Table of Contents.

Let us end this terse Chapter 1 by displaying, without any immediate explanation, 2 *rather trivial examples* of the type of N-body problems—for simplicity, with $N = 2$—the *solvability* of which, demonstrated in this book, entails an understanding of their *remarkable* phenomenology. Hence, in the rest of this short Chapter 1 we refer to 2 point particles moving in the *real* Cartesian plane, while in the rest of this book we mainly focus on N-body problems—with $N \geq 2$—characterizing the motion of N points moving in the *complex* plane.

Example 1.1. The equations of motion of the *first* of these 2 *solvable* models read as follows:

$$\ddot{\vec{r}}_n = (-1)^{n+1} \, \frac{2}{r^2} \left[\left(\dot{\vec{r}}_1 \cdot \vec{r} \right) \dot{\vec{r}}_2 + \left(\dot{\vec{r}}_2 \cdot \vec{r} \right) \dot{\vec{r}}_1 - \left(\dot{\vec{r}}_1 \cdot \dot{\vec{r}} \right) \vec{r} \right]$$
$$- (\rho_1 \, \omega)^2 \, \vec{r}_n + \left[2 \, (\rho_1)^2 - (\rho_2)^2 \right] \omega^2 \, r^{-2} \left[(r_{n+1})^2 \, \vec{r}_n - (r_n)^2 \, \vec{r}_{n+1} \right] ,$$
$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \,, \quad n = 1, 2 \mod [2] \; . \tag{1.1}$$

Here the 2-vectors $\vec{r}_1 \equiv \vec{r}_1(t)$ and $\vec{r}_2 \equiv \vec{r}_2(t)$ are the coordinates of 2 unit-mass point-particles moving in the *real* Cartesian plane as functions of the *independent* variable t ("time"); superimposed dots indicate time differentiations; the symbol $\cdot$ denotes the standard *scalar product* in the plane (so that, in particular, $r^2 = \vec{r} \cdot \vec{r}$ is the squared modulus of the 2-vector $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$); and ρ_1, ρ_2, and ω are 3 *arbitrary real nonvanishing* parameters. It is shown below that this model is *Hamiltonian* and solvable by *algebraic* operations: in fact by just finding the 2 zeros of a second-degree time-dependent polynomial explicitly known in terms of the initial data of the problem, $\vec{r}_n(0)$ and $\dot{\vec{r}}_n(0)$. Moreover—if the 2 parameters ρ_1, ρ_2 are both *rational* numbers—then this model is *isochronous*, its *generic* solutions being *all* periodic with a common period which is an *integer* multiple of the basic period $T = 2\pi / |\omega|$. Note that this model is invariant under a common multiplicative rescaling of the dependent variables $\vec{r}_n \equiv \vec{r}_n(t)$, hence it is in particular *invariant under rotations* of the Cartesian plane (as demonstrated by the covariance of its equations of motion (1.1)); while it is *not* translation-invariant. ■

Example 1.2. The equations of motion of the *second* of these 2 *solvable* models read as follows:

$$\ddot{\vec{X}} = \omega \, \dot{\hat{X}} + \rho_1 \, \omega^2 \, \vec{X} + \frac{1}{X^2} \left[2 \left(\dot{\vec{X}} \cdot \vec{X} \right) \dot{\vec{X}} - \left(\dot{\vec{X}} \cdot \dot{\vec{X}} \right) \vec{X} \right] , \tag{1.2a}$$

$$\ddot{\vec{x}} = -\frac{1}{2} \, (\rho_2 \, \omega)^2 \, \vec{x} - \frac{1}{x^2} \left\{ 2 \left(\dot{\vec{x}} \cdot \vec{x} \right) \dot{\vec{x}} - \left(\dot{\vec{x}} \cdot \dot{\vec{x}} \right) \vec{x} \right.$$
$$\left. + \left[\rho_1 + (\rho_2)^2 / 2 \right] \omega^2 \left[2 \, (\vec{X} \cdot \vec{x}) \, \vec{X} - X^2 \, \vec{x} \right] \right.$$

$$+\,\omega\left[\left(\dot{\vec{X}}\cdot\vec{x}\right)\hat{X}+\left(\vec{X}\cdot\vec{x}\right)\dot{\hat{X}}-\left(\dot{\vec{X}}\cdot\vec{X}\right)\hat{x}\right]$$
$$+\left[2\left(\dot{\vec{X}}\cdot\vec{x}\right)\dot{\vec{X}}+\left(\dot{\vec{X}}\cdot\dot{\vec{X}}\right)\vec{x}\right]\Bigg\}\,. \qquad (1.2b)$$

Here the 2 *dependent* variables $\vec{X}\equiv\vec{X}(t)\equiv(X_1(t)\,,\,X_2(t))$ and $\vec{x}\equiv\vec{x}(t)\equiv(x_1(t)\,,\,x_2(t))$ are 2-vectors describing the motion in the *real* Cartesian plane of 2 unit-mass point-particles as functions of the *independent* variable t ("time"); $\hat{X}\equiv\hat{X}(t)\equiv(-X_2(t)\,,\,X_1(t))$, and $\hat{x}\equiv\hat{x}(t)\equiv(-x_2(t)\,,\,x_1(t))$ are these vectors rotated in the Cartesian plane by $\pi/2$; $X\equiv X(t)$ and $x\equiv x(t)$ are the moduli of these 2-vectors; the symbol $\cdot$ denotes the standard *scalar product* in the plane (so that, for instance, $x^2=\vec{x}\cdot\vec{x}=\hat{x}\cdot\hat{x}$); superimposed dots indicate time-differentiations; and the 3 parameters ρ_1,ρ_2,ω are 3 *arbitrary real* parameters. Note that we used the *unambiguous* notation $\dot{\vec{X}}\cdot\dot{\vec{X}}\equiv\left(\dot{X}_1\right)^2+\left(\dot{X}_2\right)^2$ (instead of the *ambiguous* notation $\left(\dot{X}\right)^2$ that may rather be interpreted to mean $\left\{d/dt\left[(X_1)^2+(X_2)^2\right]^{1/2}\right\}^2=\left[\left(\dot{X}_1X_1+\dot{X}_2X_2\right)/X\right]^2$); and likewise for $\dot{\vec{x}}\cdot\dot{\vec{x}}\equiv(\dot{x}_1)^2+(\dot{x}_2)^2$. Also note that these equations of motion are *scale-invariant*, i.e., *invariant* under the *rescaling* transformation $X(t)\Rightarrow c\,X(t)\,,$ $x(t)\Rightarrow c\,x(t)$ with c an *arbitrary nonvanishing constant*; hence, they are *covariant*, i.e., *rotation-invariant* in the plane. And they are clearly *Newtonian*: accelerations equal forces, with the forces depending *nonlinearly* on the coordinates of the particles and on their velocities.

A *remarkable* feature of this system of 2 equations of motion (see (1.2)) is that—if the 2 parameters ρ_1 and ρ_2 are *rational* numbers,

$$\rho_n=\frac{q_n}{k_n} \qquad (1.2c)$$

where q_n and k_n are two *arbitrary coprime integers* ($q_n\neq 0$, $k_n\geq 1$, $n=1,2$), and ω is an *arbitrary nonvanishing real* parameter to which the period $T=2\pi/|\omega|$ is associated—then this model is *isochronous*: there is a *large open* set of initial data $\vec{x}(0)\,,\,\dot{\vec{x}}(0)\,,\,\vec{X}(0),\,\dot{\vec{X}}(0)$ yielding *nonsingular* evolutions that are *all periodic* with a period that is an *integer* multiple of the basic period $T=2\pi/|\omega|$. ■

These findings are merely *rather trivial quite special* examples of those discussed later in the book (see Chapter 4, where however we generally consider N-body problems with *arbitrary* $N\geq 2$). The interested reader will find their proofs there, and thereby learn how to obtain the *explicit* solutions of the initial-value problems of these 2 Newtonian 2-body systems in the plane.