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CHARACTER THEORY AND THE MCKAY CONJECTURE

The McKay conjecture is the origin of the counting conjectures in the representation theory of finite groups. This book gives a comprehensive introduction to these conjectures, while assuming minimal background knowledge. Character theory is explored in detail along the way, from the very basics to the state of the art. This includes not only older theorems but some brand new ones too. New, elegant proofs bring the reader up to date on progress in the field, leading to the final proof that if all finite simple groups satisfy the inductive McKay condition, then the McKay conjecture is true.

Open questions are presented throughout the book, and each chapter ends with a list of problems of varying degrees of difficulty.

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Character Theory and the McKay Conjecture

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Preface

It is no exaggeration to say that a new era began in the representation theory of finite groups with the formulation of the McKay conjecture. From the perspective of finite group theory itself, it had been known a long time before the McKay conjecture that the interplay between a finite group and its local subgroups was the key to, among other things, the central problem of the theory: the classification of finite simple groups. From the point of view of modular representation theory of finite groups, there were fundamental connections already established (or conjectured) between local and global structures, such as Brauer's first main theorem, Dade's cyclic defect theory, the Green correspondence, and Brauer's height zero conjecture.

In 1972, in a paper dedicated to Richard Brauer, J. McKay observed that in some simple groups the number of irreducible complex characters of odd degree was equal to the same number calculated for the group $\mathbf{N}_G(P)$, the normalizer of a Sylow 2-subgroup P of G. Of course, the groups G and $\mathbf{N}_G(P)$ are very different in general, and yet they seemed to share a fundamental invariant. If true, this was an unexplained and astonishing discovery.

A year later, I. M. Isaacs proved that McKay's observation was true for solvable groups (in some sense, the "opposite" of the class of simple groups) and for groups of odd order for every prime. Although never formally formulated in its full generality, the *McKay Conjecture* took form.

The McKay Conjecture *If G is a finite group, p is a prime and $m_p(G)$ is the number of irreducible complex characters of G of degree not divisible by p, then*

$$m_p(G) = m_p(\mathbf{N}_G(P)),$$

where $P \in \mathrm{Syl}_p(G)$.

In 1975, J. L. Alperin gave a generalization of the McKay conjecture, which involved Brauer blocks. This was also the first formal statement of a conjecture that implied the still unformulated McKay conjecture. In 1976, J. B. Olsson proved the Alperin–McKay conjecture for symmetric groups, and in 1979, T. Okuyama and M. Wajima proved it for p-solvable groups. From this point on, different families of groups were checked by many mathematicians, and soon the idea that the McKay conjecture was going to prove correct was generally accepted. As Alperin used to say informally, if instead of mathematics this were physics, the McKay model would have been adopted a long time ago.

But how could the McKay conjecture be proven? During the course of the years, several deep generalizations of the conjecture have provided hints about the ingredients of a possible proof: blocks, isometries, simplicial complexes, characters over the p-adics, derived categories; some, or all, of these should or could be involved in the proof . . . but nobody can figure out exactly how.

While searching for a general conceptual proof of the McKay conjecture, assuming such a proof really existed, the Classification of Finite Simple Groups (CFSG) was achieved, or at least announced, in 1982. It soon became clear that a possible way of proving McKay could be to use the Classification. W. Feit, in his survey on the main problems in group theory after the CFSG, listed the McKay conjecture as the first problem. At the same time, he acknowledged that this conjecture did not seem to follow from the CFSG. As is well known, many statements in group theory admit a reduction to simple groups, in the sense that they are true provided that they are checked for every simple group. It is sometimes rightly criticized, perhaps, that a proof that uses the Classification does not fully explain why the result is actually true. We agree that it is not completely satisfactory to know that a theorem holds because simple groups have a certain property, but as long as no other proof or insight is found, can one really argue against the use of the Classification? In view of so many fundamental theorems in group theory that rely on the CFSG, this is hardly debatable. As we have said, however, the McKay conjecture does not reduce to simple groups, in the sense that it is not sufficient to check the McKay conjecture for simple groups in order to have a proof for every finite group. Something more complicated is going on.

In 1987, another central conjecture in representation theory, which seemed not to be connected with the McKay conjecture, was proposed. The celebrated *Alperin's weight conjecture* (AWC) claimed that the number of non-similar irreducible representations of a finite group over an algebraically closed field of characteristic p was the same as its number of *p-weights*, which is the number of irreducible projective characters of the p-local subgroups. This wonderful conjecture by J. L. Alperin was inspired by the representation theory of groups of Lie type and followed previous related work of T. Okuyama in p-solvable groups. Two years later, a reformulation of AWC by R. Knörr and G. R.

Robinson changed the whole perspective of the problem, and a wealth of reformulations of AWC by R. Boltje, J. Thévenaz and many others showed not only that AWC lies very deep in representation theory, but that it has connections with homotopy, K-theory, and other parts of mathematics.

In 1990, inspired by the Knorr–Robinson formulation, E. C. Dade announced various versions of a far-reaching generalization of the McKay conjecture (the so-called *Dade's counting conjectures*) that he expected to reduce to simple groups. If p^e is an arbitrary power of p, Dade's conjectures predict locally how many irreducible complex characters of a finite group have degree divisible by p^e but not by p^{e+1}. Dade's conjectures imply simultaneously both the McKay conjecture and AWC, which are now part of a single statement. Unfortunately, such a reduction to simple groups never appeared in print.

Also in 1990, M. Broué proposed a deep structural categorical explanation for the Alperin–McKay conjecture, but only for abelian *defect groups*. (Broué's conjecture explains McKay's, for instance, but only for groups with abelian Sylow p-subgroups.) This conjecture generalized the celebrated cyclic defect theory, and provided new and powerful insights, even at the level of characters. Broué's conjecture has led to a great deal of research in representation theory, and at the same time has opened new areas which are far removed from character theory. It is also the only one of these global–local counting conjectures that proposes an explanation for why some global numbers can be calculated locally. It remains a challenge to find a theory that generalizes Broué's conjecture to non-abelian defect groups.

In 2007, G. Malle, I. M. Isaacs and I published a reduction of the McKay conjecture to a problem on simple groups. This reduction was specially tailored for the McKay problem, and not for the other generalizations that had already been made in the course of the years (which were making the reductions much harder). Moreover, the reduction was conducted jointly with a specialist in simple groups, Malle, in order to guarantee that what was eventually going to be asked from the simple group specialists was, conceivably, achievable. This is how the so-called *inductive McKay condition* was born. It was proved that the McKay conjecture was true for every finite group if every simple group satisfies this inductive McKay condition. This condition requires the existence of a bijection between the irreducible characters of degree not divisible by p of any quasisimple group X (a perfect central extension of a simple group) and those of the normalizer of a Sylow p-subgroup of X, that additionally is equivariant with respect to some group automorphisms and that preserves certain cohomological properties associated with the characters. Since the appearance of this paper, many simple groups have been proved to satisfy the inductive

McKay condition, and the road to proving the McKay conjecture is, in theory, now built. Also, this paper soon led to reductions of all the other counting conjectures mentioned before to statements about simple groups.

But the most spectacular success occurred in 2015, when G. Malle and B. Späth managed to check the inductive McKay condition for the prime $p = 2$, thereby proving the McKay conjecture for this prime (which was McKay's original observation). This was a confirmation that what we embarked upon in 2007 was indeed a successful path.

The inductive McKay condition has inspired research on the character theory of the groups of Lie type, putting the focus on essential open problems so as to understand how automorphisms of groups of Lie type act on their irreducible characters, and the Clifford theory associated with these.

Since 2007, there have been significant simplifications on the formulation of the inductive McKay condition on quasisimple groups, and a general theory on this, with implications for the representation theory of general finite groups, has been developed (mainly by B. Späth). We shall dedicate the last chapter of this book to this topic. We wish not only to publicize the beauty of this reduction but also to engage students in the exciting tasks that remain ahead of us.

But, of course, this book is not only about the McKay and the global–local counting conjectures. Although our main goal has been to introduce the reader to these conjectures, to explain how they are related and to show how the reduction of the McKay conjecture to a question on simple groups is conducted, in order to achieve all this we need to review many of the essential and remarkable theorems of character theory.

After we have established our basics in Chapter 1, in Chapter 2 we introduce the Glauberman correspondence, which lies at the heart of the global–local counting conjectures. In Chapter 3, we analyze Galois action on characters and we digress to discuss a conjecture of W. Feit and a related theorem of R. Brauer. In Chapter 4, we allow ourselves some more digressions, in order to present results on zeros of characters, character values, and character identities, some of which appear in book form for the first time. (Among them, is a new proof of the Brauer–Nesbitt theorem on zeros of p-defect zero characters and Knörr's characterization of them; a theorem of Strunkov on the existence of p-defect zero characters; and a theorem of Robinson on characters taking roots of unity values on 2-singular elements.) Chapter 5 on normal subgroups is fundamental, since it will give us some of the techniques needed to conduct the reductions to simple groups in several later theorems. In Chapter 6, we discuss essential criteria to extend characters, and in the first of our reduction theorems, we prove that a group of even order possesses a nontrivial rational-valued irreducible character using the CFSG. In Chapter 7, we analyze degrees of characters, complex group algebras and character tables, and also give a generalization of the Itô–Michler theorem, again relying on the CFSG. Chapter 8 is devoted to presenting a proof the Howlett–Isaacs theorem on the solution of the Iwahori–Matsumoto conjecture on the solvability of groups of central type. This is the third theorem in this book that uses the CFSG. The Howlett–Isaacs theorem also relies on more delicate properties of the Glauberman correspondence, which we develop in this chapter, and that will be used later in the reduction of the McKay conjecture. In Chapter 9 we introduce the McKay conjecture, many of its generalizations and refinements, their consequences, and the interconnections between them and the block-free forms of Alperin's weight conjecture and Dade's ordinary conjecture. In Chapter 10 we finally prove that if all finite simple groups satisfy the inductive McKay condition, then the McKay conjecture is true.

At the end of each chapter, we include a section in which we comment on some related results and open questions. Each chapter concludes with a list of problems, of varying degrees of difficulty. In the Bibliographic Notes at the end of the book, we give explicit references to some of the theorems that we have covered in the text and of relevant comments made. References to all the works mentioned in this Preface can be found in the last section of Chapter 9.

It is now time to thank some colleagues, friends, and frequent collaborators, without whom this book would not have been possible. First, I would like to thank Marty Isaacs, from whom I learned all the character theory that I know. Our collaboration and friendship started in 1989, when I visited him in Berkeley to write the first of many joint papers. This book owes him a lot. Special thanks are due to Gunter Malle, with whom we started on the road to the inductive conditions, for many helpful observations on this book, which he read from beginning to end. Also, thanks to Geoff Robinson, who has helped me in this and other projects, always providing the cleverest insights. Of course, thanks to Pham Huu Tiep for his friendship and a lasting and fruitful collaboration. Benjamin Sambale has read the whole manuscript and has given me many useful comments and corrections, as has Noelia Rizo. To both I am very grateful.

Finally, I would also like to thank Silvio Dolfi, Lucía Sanus, Britta Späth, Joan Tent, and Carolina Vallejo, who have traveled with me on parts of this journey.

Notation

$\mathrm{Char}(G)$	the set of complex characters of G
$\mathrm{Irr}(G)$	the set of irreducible complex characters of G
I_n	the identity $n \times n$ matrix
1_G	the trivial or principal character of G
$\mathrm{Lin}(G)$	the group of linear characters of G
$\mathrm{cf}(G)$	the space of complex class functions on G
$\mathrm{Cl}(G)$	the set of conjugacy classes of G
$\mathrm{Irr}(\chi)$	the set of irreducible constituents of the character χ
$X(G)$	the character table of G
$[\alpha, \beta]$	the inner product of the class functions $\alpha, \beta \in \mathrm{cf}(G)$
ρ_G	the regular character of G
$\mathrm{Mat}_n(F)$	the $n \times n$ matrices over the field F
$\mathrm{diag}(a_1, \ldots, a_n)$	the diagonal $n \times n$ matrix with a_i in the (i, i)-position
$A \otimes B$	the Kronecker product of the matrices A and B
$\mathbf{Z}(A)$	the center of the algebra A
$A \oplus B$	the direct sum of the algebras A and B
RG	the group ring of the group G over the commutative ring R
$\mathbb{C}G$	the complex group algebra
$\delta_{a,b}$	the Kronecker δ symbol
e_χ	the central primitive idempotent associated with $\chi \in \mathrm{Irr}(G)$
$\bar{\chi}$	the complex conjugate of χ; in a different context, character associated with a factor group
$\ker(\chi)$	the kernel of the character χ
$\mathbf{Z}(\chi)$	the center of the character χ
$o(\chi)$	the order of the determinant of χ
ω_χ	the central character associated with χ

$\alpha \times \beta$	the direct product of the characters α and β
$\mathbb{Z}[\mathrm{Irr}(G)]$	the ring of generalized characters of G
χ_H	the restriction of the class function χ to the subgroup H
α^G	the induced class function of α to G
θ^a	if a is an isomorphism then $\theta^a(g^a) = \theta(g)$
G_θ	the stabilizer of θ in G
$\mathrm{Irr}(G\|\theta)$	the irreducible characters of G whose restriction contains θ
$\mathrm{Char}(G\|\theta)$	the characters of G whose restriction contains θ
$\mathrm{cf}(G\|\theta)$	the $\mathbb{C}$-span of $\mathrm{Irr}(G\|\theta)$
n_p	the largest power of p dividing the integer n, or p-part of n
x_p	the p-part of the group element x
$x_{p'}$	the p'-part of the group element x
G_p	the set of elements of G whose order is a power of p
$G_{p'}$	the set of elements of G whose order is not divisible by p
$\mathrm{cl}_G(x)$	the conjugacy class of $x \in G$
$o(g)$	the order of the group element g
$\mathbf{C}_G(g)$	the centralizer of $g \in G$
$\mathbf{Z}(G)$	the center of the group G
G'	the derived or commutator subgroup $[G, G]$ of G
$\mathbf{O}_p(G)$	the largest normal p-subgroup of G
$\mathbf{O}_{p'}(G)$	the largest normal subgroup of G of order not divisible by p
$\mathbf{O}^p(G)$	the smallest normal subgroup of G whose factor group is a p-group
$\mathsf{S}_n, \mathsf{A}_n$	the symmetric and the alternating group of degree n
C_n	the cyclic group of order n
$H \rtimes K$	the semidirect product of H with K
$H \wr K$	the wreath product of H by K
$\mathbf{C}_G(A)$	the subgroup of elements of G which are fixed by A
$\mathcal{C}(G)$	the set of all chains of p-subgroups of G
$\mathcal{N}(G)$	the set of all normal chains of p-subgroups of G
$\psi^{\otimes G}$	the tensor induced character
$\mathrm{Aut}(G)_H$	the subgroup of automorphisms $\alpha \in \mathrm{Aut}(G)$ with $\alpha(H) = H$
$\mathrm{Cl}_A(G)$	the set of A-invariant conjugacy classes of G
$\mathrm{Irr}_A(G)$	the set of A-invariant irreducible characters of G
$\overline{\mathbb{Q}}$	the algebraic closure of $\mathbb{Q}$
$\mathbf{R}$	the ring of algebraic integers in $\mathbb{C}$
$\mathbb{Q}_n$	the nth cyclotomic field
$\mathbb{Q}(\chi)$	the smallest field containing the values of χ
χ^σ	the Galois conjugate character of χ by σ

$\mathrm{Gal}(F/K)$	the automorphisms of F that fix every element of K.
$\mathrm{Irr}_F(G)$	the irreducible characters of G whose values are in the field F
$\mathbb{Q}(K)$	the smallest field containing the values of all the characters on K
$\widehat{X}$	if $X \subseteq G$, this is the element $\sum_{x \in X} x$ in some group algebra
(G, N, θ)	means that $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$ is G-invariant
$\alpha(x, y)$	the factor set α evaluated in (x, y)
$\mathrm{Irr}_{p'}(G)$	the set of irreducible characters of G of degree not divisible by p
$\mathrm{Irr}_{p'}(G\|\theta)$	$\mathrm{Irr}_{p'}(G) \cap \mathrm{Irr}(G\|\theta)$
$\mathrm{Ext}(G\|\theta)$	the set of irreducible characters of G that extend θ
$\alpha \cdot \beta$	the central product of the characters α and β
$k_d(G)$	the set of irreducible characters $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)_p = \|G\|_p/p^d$