Representations of Solvable Lie Groups

The theory of unitary group representations began with finite groups, and blossomed in the twentieth century both as a natural abstraction of classical harmonic analysis, and as a tool for understanding various physical phenomena. Combining basic theory and new results, this monograph is a fresh and self-contained exposition of group representations and harmonic analysis on solvable Lie groups.

Covering a range of topics, from stratification methods for linear solvable actions in a finite-dimensional vector space, to complete proofs of essential elements of Mackey theory and a unified development of the main features of the orbit method for solvable Lie groups, the authors provide both well-known and new examples with a focus on those relevant to contemporary applications. Clear explanations of the basic theory make this an invaluable reference guide for graduate students as well as researchers.

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Basic Theory and Examples

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Dedicated to
Françoise and Julie for their support and patience.

Contents

Preface

The theory of unitary group representations began with finite groups, and blossomed in the twentieth century both as a natural abstraction of classical harmonic analysis and as a tool for understanding various physical phenomena. An important early step for abstract harmonic analysis was the theorem of Stone–von Neumann, whereby all irreducible unitary representations of the Heisenberg group are classified up to isomorphism. Subsequent work of J. Dixmier and A. A. Kirillov may be regarded as a generalization of this theorem to all simply connected nilpotent Lie groups, and ushered in the era of the method of coadjoint orbits for the more general solvable Lie groups. The list of pioneers and important results in this subject is long and distinguished; three central achievements are the Auslander–Kostant classification of irreducible unitary representations of type 1 simply connected solvable groups, the Duflo–Moore Plancherel theory, and the extension by L. Pukanszky of the Auslander–Kostant theory to non-type 1 solvable groups. A milestone in this development is the broadly influential monograph [11] by Bernat et al., in which many related results are described. During this time equally prodigious activity was devoted to the classification of irreducible unitary representations of semisimple Lie groups.

The later part of the twentieth century saw an expanded and deepened understanding of the unitary dual via the orbit method, as well as the application of the orbit method in analysis on homogeneous spaces. After the essentially exhaustive construction of factor representations, the focus shifted to properties of these representations, and in the type 1 case, to the orbital parametrization for irreducible unitary representations, as well as to the orbital decomposition of naturally occurring reducible unitary representations. Harmonic analysis also witnessed the rise of the theory of non-orthogonal expansions. Just as time-frequency analysis was found to be deeply related to harmonic analysis on the Heisenberg group, discrete wavelet analysis was found to be closely related to harmonic analysis on the affine group. In the context of Duflo–Moore theory, general methods such as co-orbit theory for discretization of unitary representations provided precise descriptions of this connection. This book arises in the context of the continuing exploration of the role of abstract harmonic analysis in contemporary applications.

The goal of this book is to develop abstract harmonic analysis in the context of a very concrete presentation of solvable Lie algebras and Lie groups and their representations. A real, connected, simply connected Lie group is completely determined by its Lie algebra, and every Lie algebra has Levi decomposition as a direct sum of a solvable ideal and a semisimple subalgebra. The theory of semisimple Lie algebras is very well known: there is a complete classification with a very precise and useful presentation, using Cartan subalgebras and roots. On the other hand, solvable Lie algebras are very far from being classified, even though the topology of the corresponding simply connected Lie group is homeomorphic with $\mathbb{R}^n$. Nevertheless, the structure of solvable Lie algebras allows for explicit embeddings into matrix algebras which, though not canonical, are useful for many purposes. Moreover, solvability makes possible a broadly successful method for explicitly describing large classes of unitary representations, including irreducible unitary representations, up to unitary equivalence. This method is based upon information about the orbits of the coadjoint representation – the canonical linear action of the group on the linear dual of its Lie algebra – and is closely related to the construction due to G. Mackey of an irreducible unitary representation – an induced representation – via the action of the group on the unitary dual of a normal subgroup. Indeed, a precise understanding of this relationship in the case where the normal subgroup is a vector group is motivated by contemporary applications of continuous and discrete wavelets.

When the group is exponential, i.e., when the exponential map from the Lie group to the Lie algebra is a global diffeomorphism, then the theory of unitary representations is understood via a canonical and explicit mapping K associating to each coadjoint orbit $\mathcal{O}$ a unitary irreducible representation $\mathrm{K}(\mathcal{O})$. By virtue of the work of J. Ludwig (see the book of Leptin and Ludwig [61]) this map is understood to be a homeomorphism between the quotient space of coadjoint orbits and the unitary dual of the Lie group with the Fell topology.

For connected, simply connected, non-exponential solvable groups, the exponential mapping is neither injective nor surjective (in fact not even everywhere regular) and irreducible unitary representations are no longer described simply by coadjoint orbits. For many such groups however, the coadjoint orbits together with additional data arising from the topology of the orbits, is sufficient to explicitly construct all irreducible unitary representations. This is the main result of [9], where Auslander and Kostant build and describe all the irreducible unitary representations of type 1 solvable groups.

However, if the group is not type 1, so that irreducible decompositions of unitary representations may not be unique or even well-defined, coadjoint orbits must be replaced by so-called generalized orbits, in order to construct factor representations. Still, for the class of type 1 solvable groups (which includes the exponential groups) the theory of irreducible unitary representations is, in some sense, simpler than for semisimple Lie groups, in that there is essentially a single construction (albeit a complicated one) that always works.

In Chapter 1, elementary results essential for solvable Lie algebras and Lie groups are presented in detail. Beginning with Lie algebras, we present fundamental definitions and results for representations of solvable Lie algebras in detail, furnishing examples along the way. When Lie groups are taken up, we briefly summarize elementary theory of Lie groups without proof, citing numerous references. Nevertheless, a number of theoretical facts specific to solvable Lie groups are proved for completeness. In particular, we recall the characterization of exponential Lie groups due to Dixmier, even though this has been presented in numerous prior sources. Important facts about finite-dimensional representations of solvable Lie groups are also proved. Examples are provided at nearly every step. In this sense the book is almost self-contained: many elements of Lie theory and representation theory are discussed in detail, and simplifications that result when restricting consideration to matrix Lie groups will suffice to account for much of the theory for which proofs are omitted. It is our hope that the book is broadly accessible and a useful tool even for beginners in Lie theory.

In Chapter 2, a stratification procedure for classifying orbits of a linear action of a solvable Lie group G on a finite-dimensional real vector space V is described in detail. This entails a suitable choice of basis for the complexification of V and a decomposition of V into finitely many G-invariant subsets called “layers,” in each of which the G-orbits are homeomorphic. If the action in a given layer is regular (that is, produces locally compact orbits) then the quotient space of orbits in the layer is Hausdorff, and for broad classes of groups can be explicitly parametrized. Since each solvable Lie group can be realized as a matrix group, application of stratification methods to the adjoint action gives a presentation of any solvable Lie group, allowing to “localize” in some sense the singularities of the action of such a Lie group. Of course such a presentation does not provide a classification, but it gives a powerful way to produce a wide variety of examples of solvable Lie algebras and Lie groups in a very explicit and useful form. Therefore, stratification methods are used throughout the book for describing examples of solvable Lie groups and their actions. In particular, when the action in question is the coadjoint action, we use our presentation to build recursively "orthogonal" elements in the Lie algebra, rationally depending upon point in a given layer. For the class of exponential groups, we present a parametrization of the quotient space of orbits in a given layer by way of explicit cross-sections, and we indicate through examples how this can be done for metabelian groups as well.

Basic facts and results about unitary representations are presented in Chapter 3. The inducing construction is developed in detail, as it is a crucial tool for subsequent results. The approach to unitary representations of solvable groups via coadjoint orbits is based upon the fundamental notion of a polarization, and the theory of polarizations is developed in detail in Chapter 4. In particular, the notion of polarization naturally leads to a generalization of the inducing construction, the resulting generalization being known as holomorphic induction; this topic is developed in detail in Chapter 5. Application of stratification results from Chapter 2 to the coadjoint representation shows that the usual construction of a polarization is in a certain sense regular in each layer, and within a given layer the associated unitary representations produced by holomorphic induction share useful properties that are used later. Chapter 5 culminates in the fundamental construction of irreducible unitary representations associated to orbit data. In Chapter 6, results of Chapter 2 are applied first to the coadjoint action. An explicit Plancherel formula is proved for exponential groups, in a form which is analogous to the well-known formula of Pukanszky for nilpotent groups (see [79]) which also appears in the book by Corwin and Greenleaf [16].

Much of the material in this book is a distillation of a massive body of research dating from the middle twentieth century. We are especially indebted to the work of prior books, [11], [40], and [16]. For nearly a half-century the monograph [11] of Bernat et al. has served as a seminal source for the theory of unitary representations of solvable Lie groups. The present volume does not intend to supplant this monograph, but rather to augment its presentation of the theory with a development that is illuminated by explicit stratification procedures and numerous examples, and to present a few of the more recent developments and applications of the theory that have appeared after its publication. The present book borrows from the efficient presentation of Mackey theory in the book [40] by G. Folland, upon which we built a self-contained exposition of necessary machinery for solvable groups. The book by L. Corwin and F. Greenleaf [16] serves in some respects as a model for the present work: the inclusion of examples at every step, the application of stratification and cross-sections for finite-dimensional representations, as well as the assumptions regarding the reader's familiarity with Lie theory, are somewhat imitated in the present text. Just as with the monograph [11], the present volume is envisioned as a supplement or companion to [16]. It is our belief that as such, this book provides a useful path to learning the basics of Lie theory and harmonic analysis in the setting of solvable Lie groups.