1

General Principles

1.1 Introduction

Magnetohydrodynamics is concerned with the study of the interaction of a fluid with a magnetic field. It takes as its philosophy a *continuum* approach, describing its phenomena in macroscopic terms rather than in terms of particle motions. Thus, it is close in spirit to fluid mechanics, which studies the properties of fluids from such a continuum viewpoint. Magnetohydrodynamics – commonly abbreviated to MHD – may also be viewed from a particle approach, discussing the motions of charged particles (electrons or ions) in the presence of a magnetic field. This is the realm of plasma physics. In our account here we will consider the subject principally from the macroscopic viewpoint.

The description of magnetic effects in magnetohydrodynamics is rooted in the celebrated equations of electromagnetism formulated by James Clerk Maxwell in 1864, though it is generally only the pre-Maxwellian form of the equations of electromagnetism that are used. The displacement current introduced by Maxwell is ignored on the basis that rapidly varying phenomena, such as electromagnetic waves, are best described from an electromagnetic viewpoint. Thus there are no electromagnetic waves or light in magnetohydrodynamics. Instead, the subject is concerned with relatively slow phenomena, such as sound waves or convective flows or field generation by dynamo action. It is to such phenomena that a fluid approach is particularly suited.

By a *fluid* we mean a gas, plasma or liquid that may be treated from a continuum approach. In magnetohydrodynamics the fluid is a conductor of electricity, and motions within the fluid occur in the presence of an applied magnetic field. The current that flows throughout the volume of the fluid is determined by Ampere's law which, when expressed in partial differential equation form, relates the current density to the 'curl' (a vector operator) of the magnetic field. Motions in the fluid are subject to a magnetic force, the Lorentz force (or $\mathbf{j} \times \mathbf{B}$ force), arising from the current density $\mathbf{j}$ and the magnetic (induction) field $\mathbf{B}$; this force, together with any others that may act (such as pressure gradients or gravity), serves to define the motion of the fluid.

Temporal changes in the magnetic field $\mathbf{B}$ are determined by Faraday's law of induction, which links such changes in $\mathbf{B}$ to the 'curl' of the electric field $\mathbf{E}$. The electric field is in turn related to the current density $\mathbf{j}$ through Ohm's law, expressed in a form appropriate for a moving conductor (the fluid). Motions within the fluid are thus inextricably linked to the magnetic field embedded within it, so that movements of the fluid entail movements in the field, and *vice versa*. It is this intimate link of fluid and field that gives magnetohydrodynamics its distinctive nature.

Magnetohydrodynamics, then, is the offspring from a marriage of fluid mechanics and electromagnetism. It was an offspring that took its time in developing. The basic physical laws and principles of its parents were well known by the end of the nineteenth century, but it was well into the twentieth century before the stirrings of magnetohydrodynamics took shape and then at first only in a somewhat sporadic fashion. By the 1940s, however, the subject was in full growth and has continued this way ever since. A brief account of the early years of development of magnetohydrodynamics is provided in Cowling (1962).

The initially slow development of magnetohydrodynamics is in some ways surprising, given the pedigree of its parents. But early experiments with laboratory fluids such as mercury or sodium, aimed at investigating magnetohydrodynamic phenomena, were fraught with difficulties, principally connected with the liquids themselves and the maintenance of sufficiently strong magnetic fields. Strong ohmic attenuation of motions made comparison between theory and experiment somewhat qualitative, though reasonable agreement was obtained. However, it was through the application of magnetohydrodynamics to large-scale phenomena, such as exhibited in the magnetic fields of the Earth and its magnetosphere, the Sun and our Galaxy, that a spur to sustained development was provided. That spur has continued to the present day, increasing to ever greater effect as space and ground-based observations of, most notably, the Sun and the Earth's magnetosphere give firm direction to magnetohydrodynamics. Moreover, the laboratory fluid has not been left behind, as detailed studies of fusion plasmas have revealed the utility of a magnetohydrodynamic description of certain phenomena as a valuable addition to a plasma approach.

An early advance in magnetohydrodynamics, though paradoxically for some time it seemed more like a backward step, was made by T. G. Cowling who showed, in 1934, that a dynamo must have a non-symmetric component (Cowling 1934). That a magnetohydrodynamic fluid could support a wave motion, distinct from the familiar electromagnetic and sound waves of the parents, was not however realized until the early 1940s. In a brief half-page letter to the journal *Nature*, H. Alfvén showed that in a perfectly conducting incompressible fluid a transverse wave may propagate along a homogeneous magnetic field; the speed of the wave was proportional to the strength of the applied magnetic field and inversely proportional to the square root of the mass density of the fluid in which the field was embedded (Alfvén 1942a, b). Alfvén termed this wave an 'electromagnetic-hydrodynamic wave', but it later became apparent that the *Alfvén wave* was born! It seems that the term 'Alfvén wave' entered into use with the work of V. C. A. Ferraro and J. W. Dungey (Ferraro 1954; Dungey 1954). For a recent general discussion of Alfvén's contribution to magnetohydrodynamic waves, see Russell (2018). Alfvén was later, in 1950, awarded the Nobel prize for his contributions to magnetohydrodynamics. In his 1942 work Alfvén made the suggestion that the observed latitudinal drift of sunspots on the Sun's surface towards the equator may be a wave phenomenon controlled by wave motions (Alfvén waves) deep below the solar surface (Alfvén, 1942a). This direct linkage of sunspot drift with Alfvén waves is not thought likely now but magnetohydrodynamic waves do arise in sunspots themselves.

Alfvén did not discuss what happens to wave motions in a fluid that is compressible. That extension, particularly important for astrophysical applications, was left to another brief letter to *Nature*, written by N. Herlofson (1950), and a more extensive treatment by H. C. van de Hulst (1951). Herlofson and van de Hulst made the remarkable theoretical discovery that in a magnetohydrodynamic medium there are in fact three modes of propagation open to the system: the Alfvén wave (uninfluenced by the compressibility of the medium) and two compressible waves. There are three distinct wave speeds associated with these modes, and moreover these speeds depend upon the *direction* of propagation of the wave. In other words, magnetohydrodynamic waves are *anisotropic*. During the 1950s and early 60s, the theoretical properties of these waves were further explored; see, for example, Friedrichs and Kranzer (1958) and Lighthill (1960).

More recently, developments in the subject of wave propagation have been motivated on a number of fronts: by the possibility of heating of laboratory plasmas by magnetohydrodynamic waves; by the observations of pulsations in the magnetosphere; by the *direct* observation of waves in the Sun's corona and their use in coronal seismology; and by the realization that astrophysical plasmas generally, but most clearly the solar atmosphere, are likely to be strongly inhomogeneous. Perhaps above all has been the spur provided by the direct observations of magnetohydrodynamic waves in the solar corona, which has undoubtedly been a powerful stimulus in the further development of theoretical aspects. Underpinning much of these theoretical developments is the detailed study of magnetohydrodynamic wave motions in structured magnetic atmospheres, which are significantly different from those of a uniform medium, though an understanding of this simpler case is, of course, the basis for any study of a non-uniform medium. In any case, structured media provide wave guides for magnetohydrodynamic waves.

We end this section with a brief comment about units. We are adopting the mks [metre kilogram second] system of units in our treatment, with electromagnetic quantities expressed in SI [System Internationale, rationalized mks] units. In magnetohydrodynamics it is convenient to regard the magnetic field $\mathbf{B}$ and the velocity $\mathbf{u}$ as the primary variables; other variables, such as the current density $\mathbf{j}$ and electric field $\mathbf{E}$, are then of secondary interest, following from a knowledge of the primary variables if and when required. Generally, then, of the electromagnetic variables, in applications or illustrations of our equations we will only quote values of the magnetic field strength $B\,(=|\mathbf{B}|)$; in SI units, B is expressed in tesla (T). However, it frequently proves convenient to quote B in gauss (G), noting that $1\,\mathrm{T} = 10^4\,\mathrm{G} = 10\,\mathrm{kG}$.

1.2 A Variety of Plasmas

Magnetohydrodynamics has found application to a wide variety of plasmas, extending from the small scale of the laboratory plasma to the vast scale of the galactic medium. The fact that the Earth – and indeed many of the planets – has a magnetic field has prompted the development of magnetohydrodynamic theories aimed at describing its maintenance and temporal variation. This is the magnetohydrodynamic dynamo problem (see, for example, Moffatt 1978; Parker 1979a). In particular, extensive developments have taken place in connection with the Sun's plasma: in its interior (where magnetic fields are both stored and manipulated) through to its surface (where fields are observed and measured in great detail); into the tenuous but enigmatic coronal outer atmosphere; and on into the solar wind that blows past the Earth and the other planets, interacting with their magnetic fields. The magnetospheres that envelop the magnetized planets are ever subject to variations in the solar wind, variations that commonly have their origin in events in the Sun's lower atmosphere. The Earth's magnetosphere, in particular, displays an array of phenomena that may be modelled using MHD and exhibits a variety of oscillations (Walker 2005; Wright and Mann 2006; Southwood, Cowley and Mitton 2015). Indeed, it is profitable to compare and contrast oscillatory phenomena in the Earth's magnetosphere and in the Sun's atmosphere (Nakariakov *et al.* 2016). However, our chief interest here is the solar atmosphere. Accordingly, we turn now to a brief overview of the Sun.

1.2.1 The Sun

The most distinctive property of the Sun as a plasma is its size. With a radius of $R_\odot = 6.96 \times 10^8$ m, the Sun displays a wide variety of plasma conditions ranging from its hot and dense interior, out through its visible and relatively cool surface, and on into its hot but tenuous atmosphere. Gravitational stratification makes for a complicated plasma, doubly compounded by the fact that the Sun possesses a complex and often dynamic magnetic field. Magnetism is the cause of almost all the exotic phenomena displayed by the Sun; for without a magnetic field the Sun would be a very quiet and relatively uninteresting plasma indeed. The possession of a magnetic field is a property it shares with a wide range of stars, many of which must surely display yet more exotic phenomena than we see on the Sun, simply by virtue of their stronger magnetic fields; for to detect that a star has a magnetic field, that field must be about 10^2 times stronger than occurs in the Sun viewed as a star.

The Sun's size and the variety of phenomena it displays has led us to regard the plasma as made up of separate regions. This is a convenient view to take, though one should not overlook the fact that these different regions are connected to one another. The Sun's interior, the region below the visible surface, is divided into three zones: an inner core, where nuclear reactions maintain the heat supply; a radiative zone, where the generated heat is distributed outwards by radiative transport; and, occupying the immediate layers below the visible surface, a *convection zone*, where heat transport is in the form of convective cells. Blending in with the top of the convection zone is the *photosphere*, the visible layer of the solar surface. The photosphere extends upwards for a height of about 500 km by which the Sun's temperature has fallen to its lowest value, of about 4200 K. This is the *temperature minimum*. Higher still in the atmosphere, the temperature rises, at first slowly in the *chromosphere* but then rapidly as we enter the *corona*. The temperature of the chromosphere ranges from the temperature minimum value to some 5×10^4 K whereafter it rises steeply in a thin layer, known as the *transition region*, to the order of 10^6 K. This hot outer region is known as the *corona*. Gravitational stratification ensures that the plasma density falls off with height, so the chromosphere and more especially the corona are tenuous plasmas in comparison with the photosphere.

Both the chromosphere and the corona are dominated by magnetism. That the Sun has a magnetic field was not known until 1908, when G. E. Hale used the then newly discovered Zeeman effect to measure the magnetic field of sunspots (Hale 1908), the dark blemishes frequently visible on the solar surface (the photosphere). Hale obtained field strengths of typically 3000 G. The field is sufficiently strong and covers a sufficiently large region – about 10^4 km across – that it locally modifies the convective transport of heat, resulting in a cool patch in the photosphere.

Sunspots are magnetically complex structures. They are frequently observed in groups, where their magnetism tends to blend together to produce complex field patterns in the solar atmosphere that are somewhat similar to the patterns made by iron filings one sees around bar magnets in the school laboratory. But even in an isolated spot the field patterns detectable in the photosphere and chromosphere are complex. There are two distinct regions of a mature spot: its central cool *umbra* where the field is strongest and is predominantly vertical, and a surrounding *penumbra* where the field is weaker and has bent over towards the horizontal. The temperature in the umbra is typically 4000 K, compared with about 5000 K in the penumbra and 6000 K in the photosphere.

The magnetism measured in sunspots at the solar surface is presumed to be generated and manipulated by flows deep within the interior. The solar interior, made up of about 90% hydrogen and 10% helium (there are also small amounts of heavier elements), is believed to consist of an inner core where nuclear reactions keep the plasma exceedingly hot, at some 1.6×10^7 K. The inner core occupies some 25% of the solar radius. The region outside the inner core and extending out to about 70% of the solar radius is the radiative zone, a region where energy transport is predominantly by radiation. The outer 30% of the solar interior is occupied by the convection zone, a region where convective cells carry the heat at the bottom of this zone out to the cooler solar surface some 2×10^8 m above. There are several scales of convection operating. The two most distinctive convective patterns of flow are the supergranules and the granules. Granules have horizontal sizes ranging between 200 km and 2000 km, with 1000 km providing a characteristic scale. The flows in granules are fairly vigorous, at some $1-3\,\mathrm{km\,s^{-1}}$ (about 2000−6000 miles per hour), to be compared with terrestrial wind speeds in hurricanes of perhaps 150 miles per hour and the Earth's record wind speed of 230 miles per hour recorded on the top of Mt Washington in the USA. Supergranules, with a horizontal scale of about 3×10^4 km and so typically 30 times bigger than the granules, have flows of $0.1-0.4\,\mathrm{km\,s^{-1}}$. Both these flow patterns are detectable in the Sun's surface layers, with the granules enveloped by the supergranules. It is at the base of the convection zone and just below that magnetic field lines are believed to be manipulated by Coriolis forces and brought to the solar surface through buoyancy effects.

Sunspots are the obvious locations of magnetism in the solar surface. But even away from sunspots there are smaller-scale concentrations of magnetic field. The smallest of these concentrations are the *intense magnetic flux tubes*, which typically occupy regions about 200 km across wherein magnetic fields of some 1−2 kG strength are confined by external gas pressure forces. The intense tubes are generally located in the regions between convective cells where downdraughts occur. Outside of sunspots, over 90% of the magnetic flux appearing in the Sun's surface layers is in the form of concentrated flux tubes. Sunspots are known to support several types of wave motion.

Above the photosphere the concentrations of magnetic field, be they in the small-scale intense flux tubes or the larger scale sunspots, rapidly spread out to fill the available space. This is simply a consequence of stratification. At the photospheric level the gas pressure is sufficient to confine the magnetic fields, once formed in a concentrated form. But the confining external pressure falls off exponentially fast from the photosphere to the chromosphere, decreasing by a factor of e (= 2.178 . . .), to some 37% of its value, over a distance of about 150 km, and this permits the confined magnetic fields to expand out in immediate response. By the mid-chromosphere the fields have filled the atmosphere and at coronal levels completely dominate the nature of the plasma.

The coronal plasma is characterized by its low density – in terrestrial terms it would be regarded as almost a vacuum – and high temperature. The high temperature of the corona, in excess of 10^6 K, was discovered in the 1930s and it remains one of the great puzzles of solar physics: what effects conspire to reverse the strong decline in temperature from the interior of the Sun to its surface, producing an extremely hot outer atmosphere? The answer to this question is important not only for the Sun but for stellar physics in general, for a wide variety of stars are believed to possess a corona.

Observations of the Sun's corona from space have revealed that in X-ray and EUV wavelengths the corona appears not as an amorphous hot glow, as was commonly thought prior to the Skylab mission in the 1970s, but as a complex and *structured* atmosphere; for an extensive discussion see Aschwanden (2004) and Priest (2014). The basis for this structure is the ubiquitous presence of magnetism in the corona. Despite the overall complexity of the coronal plasma, it would appear that there are fundamentally two different coronae: regions in which the magnetic field lines are curved in the form of loops or arcades with their ends anchored in the dense photosphere, and regions where the field lines emanate from the photosphere but are then carried out into space. The regions with re-entrant magnetic fields – the magnetic loops – are the hottest and most dense parts of the corona; they glow the brightest in X-ray and EUV pictures of the Sun. These are the *active regions*. They are characterized by temperatures of $2-3 \times 10^6$ K, plasma densities of 10^{16} particles per m^3 and have magnetic field strengths of about 10^2 G. The magnetic field, with its footpoints tied to the photosphere, confines the coronal plasma and heats it to its high temperature. By contrast, where the field is open the plasma blows out into space; these are the *coronal holes*, the source regions of the high-speed solar wind that blows from the Sun and flows on past the Earth. The plasma density in coronal holes, at 10^{14} particles per m^3, is two orders of magnitude less dense than that in the active regions. The plasma is also cooler, at $1.5-2 \times 10^6$ K, and the open magnetic field, with a strength of around 10 G, is a factor of 10 weaker than the field in active regions. These, then, are the two fundamentally different regions of the corona. Of particular interest here is the discovery by the space instrument *TRACE* (Transition Region And Coronal Explorer) that coronal loops support a variety of oscillations. Oscillations carry information about the medium in which they occur; such information may be used to obtain indirectly solar quantities that are otherwise difficult to measure. This is the new subject of *coronal seismology*.

There are, of course, many other structures in the corona in addition to the two fundamental forms. Of particular interest are *quiescent prominences*. In a sense, quiescent prominences are bits of the chromosphere that find themselves in a coronal environment. They are cool, dense structures, sometimes resembling a thin sheet of dense plasma, magically suspended in a tenuous corona. The source for their support is the magnetic field that threads through the prominence. They are generally passive structures, surviving for long periods (perhaps months) but then dramatically erupting, only to reform in much the same location shortly thereafter. In photographs of the chromosphere and corona they show up as thin (perhaps 6000 km across), filament-like, dark curves winding their way (for some 2×10^4 km) through the local magnetic structure; their height is about 5×10^4 km. Prominences have a typical density of $10^{17}\,\mathrm{m}^{-3}$, some two orders of magnitude larger than in their coronal surroundings, and a typical temperature of 7000 K. (There is evidence that the corona may be locally somewhat depleted in density in the neighbourhood of a prominence.) Quiescent prominences are observed to oscillate, a fact which may have important implications for *coronal and prominence seismology*.

1.3 The Magnetohydrodynamic Equations

We have remarked above that magnetohydrodynamics is a combination of fluid mechanics and electromagnetism with Maxwell's displacement current neglected. Here we describe the equations of this subject. We do not provide a derivation of these equations from basic principles; that route has been fully described elsewhere. Instead, we prefer to simply write down each of the relevant equations and to add some explanatory comments to illustrate various features of the equations. Derivations and discussions of the properties of the equations are given in, for example, Alfvén (1950), Cowling (1957, 1976), Kendall and Plumpton (1964), Ferraro & Plumpton (1966), Jeffrey (1966), P. H. Roberts (1967), Boyd and Sanderson (1969, 2003), Parker (1979a, 2007) and Priest (1982, 2014). Solar applications are given special attention in Bray and Loughhead (1974), Parker (1979a), Priest (1982, 2014), Bray *et al.* (1991), Choudhuri (1998), Goossens (2003), Aschwanden (2004), Goedbloed and Poedts (2004), Goedbloed, Keppens and Poedts (2010), Narayanan (2013), Ryutova (2015) and Nakariakov *et al.* (2016).

Consider a fluid with mass density ρ and motions $\mathbf{u}$. Conservation of matter – the statement that matter is neither created nor destroyed within the system (so that there are no sources or sinks of matter) – is described by the equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0. \tag{1.1}$$

Equation (1.1) is commonly referred to as the equation of continuity.

The equation of momentum is the statement that changes in momentum are a result of forces acting in the fluid; it is Newton's second law applied to a fluid. The momentum equation is

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \operatorname{grad})\mathbf{u} \right] = -\operatorname{grad} p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} + \mathbf{F}. \tag{1.2}$$

Here p denotes the fluid (or plasma) pressure and $\rho\mathbf{g}$ is the force per unit volume on the fluid because of gravity of vectorial strength $\mathbf{g}$ (we will generally assume gravity to be uniform). $\mathbf{B}$ denotes the magnetic field that threads the fluid and $\mathbf{j}$ is the current density; these two terms produce the magnetic body force $\mathbf{j} \times \mathbf{B}$, perpendicular to both $\mathbf{B}$ and $\mathbf{j}$. Finally, there may also be other forces $\mathbf{F}$ acting, such as the viscous force.

The magnetic field $\mathbf{B}$ is related to the current density $\mathbf{j}$ by Ampere's law, namely

$$\mu\mathbf{j} = \operatorname{curl}\mathbf{B}, \tag{1.3}$$

where μ is the magnetic permeability of the fluid; generally it is assumed that $\mu = 4\pi \times 10^{-7}$ henry m^{-1}, its value in free space.

Temporal changes in the magnetic field $\mathbf{B}$ are related to spatial changes in the electric field $\mathbf{E}$ through Faraday's law of induction:

$$\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{E}. \tag{1.4}$$

There is a constraint on the magnetic field: it must be solenoidal,

$$\operatorname{div}\mathbf{B} = 0. \tag{1.5}$$

This constraint is the statement that there are no magnetic monopoles: magnetic field lines have no ends, but either close upon themselves or are infinite in extent (which we may view as closing at infinity). There is thus a sharp contrast between magnetic field lines and electric field lines, for the latter originate in concentrations of charge and so may be viewed as emanating from a point.

In view of the vector identity

$$\operatorname{div}\operatorname{curl} \equiv 0,$$

we see that equation (1.4) implies that $\partial(\operatorname{div}\mathbf{B})/\partial t = 0$ and so, as a consequence of Faraday's law of induction, $\operatorname{div}\mathbf{B}$ is time independent (and thus is zero for all times if zero at any instant). The constraint (1.5) is stronger, though, insisting that the divergence of $\mathbf{B}$ is necessarily zero always. There is also an implied constraint on lines of current $\mathbf{j}$, for the above vector identity taken with Ampere's law (1.3) implies that $\operatorname{div}\mathbf{j} = 0$, and so lines of current density $\mathbf{j}$ (like lines of magnetic field) also have no ends.

The electric field $\mathbf{E}$ is related to the current density $\mathbf{j}$ by Ohm's law, as applied to a moving conductor – the fluid moving with an internal velocity $\mathbf{u}$:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \tag{1.6}$$

where σ is the electrical conductivity of the fluid. Here $\mathbf{E} + \mathbf{u} \times \mathbf{B}$ is the total electric field in the fluid, allowing for the induced electric field arising from the component of motion $\mathbf{u}$ across the field $\mathbf{B}$.

By combining equations (1.4) and (1.6) we may eliminate the electric field $\mathbf{E}$:

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u} \times \mathbf{B}) - \operatorname{curl}\left(\frac{1}{\sigma}\mathbf{j}\right). \tag{1.7}$$

Using Ampere's law (1.3), we may eliminate **j** to obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{u} \times \mathbf{B}) - \mathrm{curl}(\eta\, \mathrm{curl}\, \mathbf{B}), \tag{1.8}$$

where we have written $\eta = 1/(\mu\sigma)$; η is referred to as the magnetic diffusivity of the fluid, and has units $\mathrm{m^2\,s^{-1}}$. Equation (1.8) is the general form of the magnetohydrodynamic induction equation.

Changes in the fluid are generally considered to proceed according to an energy balance equation of the form

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, p = \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, \rho\right) - (\gamma - 1)\mathcal{L}, \tag{1.9}$$

where $\mathcal{L}$ is the gain or loss function (energy per unit volume) and γ is the ratio of specific heats at constant pressure and constant volume. The term $\mathcal{L}$ includes contributions from thermal conduction and radiation. Mechanical heating from external sources as well as the *Joule* (or *ohmic*) *heating* may also be added to the right-hand side of (1.9). Joule heating, arising from the dissipation of current within the fluid, amounts to j^2/σ watts $\mathrm{m^{-3}}$, for a current density of strength $j\ (= |\mathbf{j}|)$. Frequently all these heat losses or gains are considered to be negligible, and then isentropic (or adiabatic) conditions pertain:

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, p = \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, \rho\right). \tag{1.10}$$

The ratio of specific heats, γ, is generally assumed to be constant. In numerical illustrations we take $\gamma = 5/3$, the value appropriate for a fully ionized gas. Heat losses are discussed in Chapter 12.

The fluid we are considering will be treated as a perfect gas, for which the ideal gas law is

$$p = \frac{k_B}{\hat{m}}\rho T, \tag{1.11}$$

where $k_B\ (= 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}})$ is the Boltzmann constant, T is the absolute temperature of the fluid in degrees kelvin (K), equal to the temperature in degrees celsius (°C) plus 273, and $\hat{m}$ is its mean particle mass.

1.4 Some Properties of the MHD Equations

The above system of equations forms the basis for a description of waves in a magnetohydrodynamic fluid. However, as the physicist E. N. Parker says in the Preface of his 1979 monograph *Cosmical Magnetic Fields*, treating the physics of large-scale magnetic fields in fluids, 'The fundamental equations of physics may contain all knowledge, but they are close-mouthed and do not volunteer that knowledge' (Parker 1979a). Thus, in particular, the nature of wave propagation in a magnetic fluid, as described by the equations introduced earlier, is not transparent and indeed serves as the topic for this book. Certain basic features of the equations can, however, be immediately uncovered and these act as points of illumination in our general discourse. We set out these aspects here, as a preliminary to our more detailed discussion of the nature of wave propagation.

1.4.1 Fundamental Speeds

The magnetohydrodynamic equations have embedded within them the usual equations of acoustics, which follow by taking $\mathbf{B} \equiv 0$. Consequently, the magnetohydrodynamic equations must contain the familiar sound speed c_s, defined by

$$c_s = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}. \tag{1.12}$$

Here ρ_0 and p_0 refer to the fluid density and pressure in the unperturbed state of the medium. The occurrence of such a speed as (1.12) is evident on general dimensional grounds. For, by balancing in the momentum equation (1.2) the acceleration term $\rho(\partial \mathbf{u}/\partial t)$ with the pressure force grad p we obtain the dimensional combination

$$V\tau^{-1} \sim \frac{p}{L\rho}.$$

Here V denotes a characteristic speed, L a characteristic length, τ a characteristic timescale, with p and ρ denoting a characteristic pressure and density; the use of '$\sim$' here denotes a dimensional balance. Writing $V \sim L\tau^{-1}$ then gives $V^2 \sim p/\rho$, which leads to the combination (1.12) for a characteristic speed (though without the important factor of γ) when we take the equilibrium pressure p_0 and density ρ_0 as representative values.

A similar argument for a magnetic speed can be made by equating, in dimensional terms, the acceleration term $\rho(\partial \mathbf{u}/\partial t)$ with the magnetic force $\mathbf{j} \times \mathbf{B}$. We obtain $\rho V \tau^{-1} \sim JB$, where J and B are characteristic values of the current density and magnetic field. But from Ampere's law (1.3) we have $\mu J \sim BL^{-1}$, which allows us to eliminate J. Setting $V \sim L\tau^{-1}$ then gives $V^2 \sim B^2/(\mu\rho)$. We thus obtain a characteristic magnetic speed, a speed that arises in phenomena for which the magnetic force plays a role. We take this speed[1] as

$$c_A = \left(\frac{B_0^2}{\mu \rho_0}\right)^{1/2}, \tag{1.13}$$

choosing an equilibrium field strength B_0 and a plasma density ρ_0 as representative values of the magnetic field and fluid density. The speed c_A defined by equation (1.13) is the *Alfvén speed*, the speed obtained by Alfvén in his short letter to *Nature* in 1942 (Alfvén 1942a). It is central to all magnetohydrodynamic wave phenomena, just as the sound speed is central to all acoustic phenomena.

The sound speed and the Alfvén speed underpin all wave phenomena described by the magnetohydrodynamic equations. Other speeds also play an important role, but these are always constructed in terms of c_s and c_A. Accordingly, we consider the sound and Alfvén speeds in a little more detail.

Plasma Pressure and Density

To begin with suppose our medium is a fully ionized hydrogen plasma, consisting of n_e electrons and n_p protons (ions) in each unit volume of space. The total pressure is

$$p = n_e k_B T_e + n_p k_B T_p$$

[1] In the Gaussian cgs system of units, the Alfvén speed is define as $c_A = B_0/(4\pi\rho_0)^{1/2}$ where the magnetic field strength B_0 is in gauss (G) and the plasma density ρ_0 is in grams per cubic centimetre (g cm^{-3}).