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Quasi-Hopf Algebras

A Categorical Approach

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Preface

Some basic ideas in mathematics are very generic and almost omnipresent. Let us just mention "operators on some structure," an idea going back to symmetry of geometric configurations, and also "duality." These ideas are also at the roots of the modern theory of quasi-Hopf algebras, which is the topic of this book.

Geometry is at the root of many developments in mathematics, and for our topic of interest we may go back to algebraic geometry and the theory of (affine) algebraic varieties, which may be seen as sets of solutions of polynomial equations in some affine space over some field. One then studies such varieties via the ring of functions on them with values in the base field; in fact one restricts attention to polynomial functions forming the coordinate ring of the variety. There one observes the fundamental duality between commutative (affine) algebra and the algebraic geometry of (affine) algebraic varieties, later better phrased in the more general scheme theory.

The other generic idea of operators acting on geometric structures led directly to actions or transformation groups and operator algebras. The idea of group actions and their invariants is deeply embedded in the philosophy of mathematics; for example, in the "Erlangen Program" of F. Klein, geometry was redefined as the study of properties invariant for actions of transformation groups. On the more algebraic side, actions of groups of automorphisms of fields were used by E. Galois to solve some problems about solutions to polynomial equations over a field. In the resulting Galois theory another duality appeared, namely the duality between subgroups of the Galois group of some field extension and the lattice of subfields of the field. This Galois duality originally was considered for finite-dimensional separable field extensions but it was extended to inseparable extensions by using derivations and higher derivations, leading to Lie algebra actions and their invariants. Thus, a more general Galois theory mixing Lie actions (of derivations) and group actions (of automorphisms) resulted, immediately leading to a Galois theory for Hopf algebra actions. Further extensions of the Galois theory were in the direction of continuous groups, later called Lie groups. So here the generic ideas of action and duality met, and Hopf algebras appeared naturally. But also the geometric line of development showed a similar phenomenon with the study of abelian varieties and algebraic groups. Roughly stated, an algebraic group is an algebraic variety with a group structure on its points;

interesting examples are matrix groups, that is, groups embedded in a matrix ring and having the structure of an algebraic variety, like $\mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k)$, the general and special linear groups over the field k, respectively. The group structure on the variety translates into a structure of the coordinate ring given by a comultiplication, a counit and an antipode satisfying suitable conditions that turn it into a commutative Hopf algebra. Hopf algebras got their name because they appeared first in a celebrated paper by H. Hopf on algebraic topology. In fact the structure was discovered on the cohomology ring of an H-space; roughly stated, that is a topological space with a multiplication on it together with a special element such that left and right multiplication by this element defines a map which is homotopic to the identity map (so a kind of neutral element up to homotopy).

Group actions on vector spaces may be studied by looking at modules over the group algebra $k[G]$ of the acting group G over the base field k; similarly, Lie algebra actions of a Lie algebra $\mathfrak{g}$ on a vector space may be studied by looking at the universal enveloping algebra of $\mathfrak{g}$ over k, say $U_k(\mathfrak{g})$. Now both $k[G]$ and $U_k(\mathfrak{g})$ are Hopf algebras but not commutative anymore; instead, they are cocommutative. So aspects of group actions and Lie algebra actions become unified in a theory of actions of arbitrary Hopf algebras on general algebras or vector spaces or modules, and this received extensive interest in ring theory.

Let us point out one important "generality" for general Hopf algebras: they need not be commutative or cocommutative, as many of the early examples of Hopf algebras were. In his famous address to the International Congress of Mathematicians in 1986, Drinfeld introduced the term "quantum group," roughly referring to a quasitriangular Hopf algebra, that is, a Hopf algebra endowed with a so-called R-matrix, satisfying certain axioms that represent a relaxation of the cocommutativity condition and implying the (equally famous) quantum Yang–Baxter equation. Drinfeld proved that any finite-dimensional Hopf algebra can be embedded in a quasitriangular one, called its quantum (or Drinfeld) double. There is a vast literature on quantum groups and many examples could be obtained from deforming well-known easier Hopf algebras. Combined with the restriction to special Hopf algebras it also makes sense to restrict to special categories of modules like so-called Yetter–Drinfeld modules, to name just one.

Essential for the transition from Hopf algebras to quasi-Hopf algebras was the concept of monoidal category, roughly stated a category with a product (called the "tensor product") generalizing the tensor product of vector spaces in a suitable way and satisfying natural conditions. For example, the category of sets is a monoidal category, the "tensor product" being the Cartesian product of sets. One of the axioms of a monoidal category is the so-called "associativity constraint," which for the categories of vector spaces and of sets is "trivial;" for instance, for vector spaces this boils down to saying that, if U, V, W are vector spaces, then $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ can be identified in the usual (or "trivial") way.

One of the fundamental features of a Hopf algebra, H, is that its category of (left) representations is a monoidal category, with tensor product inherited from the category of vector spaces, and the tensor product of two left H-modules is again a left H-module via the comultiplication of H. The associativity constraint is, again, "trivial."

If one is not interested in an a priori given type of algebra but wants to make sure that there is a "product" on the category of its representations, then one finds the motivation for the introduction of quasi-Hopf algebras as Drinfeld did in his seminal paper [80]. Roughly, a quasi-Hopf algebra is an algebra for which its category of left modules is monoidal, but maybe with non-trivial associativity constraint. More precisely, what Drinfeld did was to weaken the coassociativity condition for a Hopf algebra so that the comultiplication is only coassociative up to conjugation by an invertible element of $H \otimes H \otimes H$ (which is a sort of 3-cocycle). Moreover, examples of quasi-Hopf algebras can be obtained by "twisting" the comultiplication of a Hopf algebra via a so-called "gauge transformation" (only if the gauge transformation is a sort of 2-cocycle is the twisted object again a Hopf algebra). After specialization to quantum groups, sometimes just taken to be non-commutative non-cocommutative Hopf algebras but usually with extra conditions like quasitriangularity, the generalization in terms of non-coassociativity became popular too and it found several applications as well. Again, the fundamental property is that the relaxation of coassociativity still makes the representation category into a monoidal category, and moreover the rigidity (i.e. the existence of dual objects) of the category of finite-dimensional representations of a Hopf algebra, owing to the presence of an antipode, is preserved by replacing the notion of an antipode by a suitable analogue. Categorically speaking, passing from the category of Hopf algebras to the one of quasi-Hopf algebras does not (in principle) really add to the complexity; in fact the latter is in some sense more manageable because of the presence of a kind of gauge group.

Monoidal categories were present, if hidden, in the classical ideas mentioned before and they have been very useful in obtaining a unified theory. One of the early facts that stimulated interest in monoidal categories stemmed from their applicability in rational conformal field theory (RCFT). The monoidal categories in RCFT could, by Tannaka–Krein reconstruction, be considered as module categories over some "Hopf-like" algebras. Back in 1984 Drinfeld and Jimbo introduced a quantum group by deforming a universal enveloping algebra $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$; in fact for every semisimple Lie algebra they constructed what was called afterwards the Drinfeld–Jimbo algebra. For the study of some categories of modules over the Drinfeld–Jimbo algebras, a relation with the so-called KZ-equations had to be used; these equations were introduced by Knizhnik and Zamolodchikov in 1984. The KZ-equations are linear differential equations satisfied by two-dimensional conformal field theories associated with affine Lie algebras. Such KZ-equations may be used to obtain a quantization of universal enveloping algebras, and Drinfeld used KZ-equations to construct a quasi-Hopf algebra for some Lie algebra $\mathfrak{g}$, say $Q_{\mathfrak{g}}$, so that some categories of modules over $Q_{\mathfrak{g}}$ are equivalent to similar ones over the Drinfeld–Jimbo algebra of the Lie algebra $\mathfrak{g}$. Further interesting applications of KZ-equations follow, for example, from the fact that their monodromy along closed paths yields a representation of the braid group. We refer to the specialized literature for more detail concerning applications in physics. We do the same for some deep relations with number theory in the sense of A. Grothendieck's "Esquisse."

In this book we aim to develop the theory of quasi-Hopf algebras from scratch, or almost, dealing mainly with algebraic methods. Knowledge of Hopf algebras will benefit the reader but we do introduce the necessary concepts. Using monoidal categories as the main tool makes for a rather abstract treatment of the material, but we hope the unifying effect of it will expose well the beautiful generalization from Hopf algebras to quasi-Hopf algebras; moreover, the categorical point of view also stays close to the applications in physics, as indicated by the foregoing remarks.

We now outline the content of the book (more historical and bibliographical remarks can be found in the Notes section at the end of each chapter).

In Chapters 1 and 2 we present the basic categorical concepts and tools needed for the rest of the book (monoidal, rigid and braided categories and algebras, coalgebras and Hopf algebras in such categories). We included detailed definitions and proofs; we do not assume that the reader has prior knowledge of these topics. In particular, we introduce the concepts of coalgebra, bialgebra and Hopf algebra in the usual sense (over a field), so we do not assume from the reader a knowledge of these concepts either.

In Chapter 3 we introduce the main objects of our study, quasi-bialgebras and quasi-Hopf algebras (as well as the dual concepts), present their basic properties and some classes of examples. We have two warnings for the reader: (1) the concept of quasi-bialgebra is introduced in Definition 3.4, but afterwards we make a reduction, and the axioms of a quasi-bialgebra that will be used from there on are the ones presented in equations (3.1.7)–(3.1.10); (2) unlike Drinfeld, we do not include the bijectivity of the antipode in the definition of a quasi-Hopf algebra, and we shall see in later chapters that the bijectivity is automatic in the finite-dimensional and the quasitriangular case.

In Chapter 4 we study "(co)actions" of quasi-bialgebras and quasi-Hopf algebras, namely we introduce the concepts of module (co)algebra and (bi)comodule algebra over a quasi-bialgebra, we give some examples and present some connections that exist between these structures.

In Chapter 5 we introduce various types of crossed products that appear in the context of quasi-Hopf algebras (smash products, diagonal crossed products, etc.), we study the relations between them and as an application we present a duality theorem for finite-dimensional quasi-Hopf algebras.

In Chapter 6 we introduce so-called quasi-Hopf bimodules over a quasi-Hopf algebra H, prove some structure theorems for them leading to the fact that their category is monoidally equivalent to the category of left H-modules and, as an application, we prove a structure theorem for quasi-Hopf comodule algebras.

In Chapter 7 we study finite-dimensional quasi-Hopf algebras, more precisely integrals and cointegrals for them. We use the machinery provided by Frobenius algebras, and we present some basic results about Frobenius, symmetric and Frobenius augmented algebras (so again we do not assume from the reader a knowledge of these topics). A consequence of the theory we develop is that the antipode of a finite-dimensional quasi-Hopf algebra is bijective. We end the chapter with a section containing a freeness result for quasi-Hopf algebras (for that section the reader is assumed to have some knowledge of module theory).

In Chapter 8 we introduce the four categories of Yetter–Drinfeld modules over a quasi-Hopf algebra, prove that they are all braided isomorphic and, when restricted to finite-dimensional objects, rigid. Then we introduce the quantum double of a finite-dimensional quasi-Hopf algebra (for the moment, only as a quasi-Hopf algebra), and two particular cases, objects denoted by $D^{\omega}(H)$ and $D^{\omega}(G)$ (for the latter, G is a finite group and $D^{\omega}(G)$ is called the twisted quantum double of G). We end the chapter with some properties and examples of algebras in Yetter–Drinfeld categories.

In Chapter 9 we define so-called two-sided two-cosided Hopf modules over a quasi-Hopf algebra, prove that their category is monoidally equivalent to a category of Yetter–Drinfeld modules and use this equivalence to prove some structure theorems for bicomodule algebras and bimodule coalgebras. We characterize Hopf algebras within the category of two-sided two-cosided Hopf modules and use this to define biproduct quasi-Hopf algebras.

In Chapter 10 we study quasitriangular quasi-Hopf algebras, QT for short. We show that the antipode of a QT quasi-Hopf algebra is inner, hence bijective. We prove that the quantum double of a finite-dimensional quasi-Hopf algebra is a QT quasi-Hopf algebra and we characterize the quantum double of a QT finite-dimensional quasi-Hopf algebra as a certain biproduct quasi-Hopf algebra.

In Chapter 11 we introduce the concept of factorizable quasi-Hopf algebra, prove that the quantum double of a finite-dimensional quasi-Hopf algebra is factorizable, and describe the quantum double of a factorizable quasi-Hopf algebra. We prove also that any factorizable quasi-Hopf algebra is unimodular (i.e. the spaces of left and right integrals coincide).

In Chapter 12 we describe the integrals of a quantum double of a finite-dimensional quasi-Hopf algebra (reproving that it is unimodular). We define the quantum dimension of an object in a braided rigid category, apply this to the category of finite-dimensional modules over a quasi-Hopf algebra and compute the quantum dimension of a finite-dimensional quasi-Hopf algebra H and of its quantum double $D(H)$ regarded as left $D(H)$-modules. We present a trace formula for quasi-Hopf algebras, and then we introduce the concept of involutory quasi-Hopf algebra.

In Chapter 13 we introduce the concepts of balanced and ribbon categories, leading to the concept of ribbon quasi-Hopf algebra, which is a QT quasi-Hopf algebra endowed with an element (called ribbon element) satisfying some axioms. In the final two sections, we present two classes of examples of ribbon quasi-Hopf algebras.

We have tried to make this book as self-contained as possible, providing as many details (in definitions and proofs) as we could. Owing to lack of space, we had to leave aside some other topics on quasi-Hopf algebras that would have deserved to be presented here (we intentionally left aside Drinfeld's theory of quantum enveloping algebras, because this is very well presented in C. Kassel's book [127]). In order to help the reader to get an idea of what else can be said about quasi-Hopf algebras, we have included in the bibliography a number of papers on (or related to) quasi-Hopf algebras that we did not cite or use in the book. We have also included some papers or books about Hopf algebras or category theory or other topics that we considered relevant for us or for the subject of the book.

This book is an outcome of the long-term scientific cooperation between the Non-commutative Algebra groups from Antwerp, Brussels and Bucharest. We wholeheartedly thank our colleagues from the University of Antwerp, the University of Brusssels, the University of Bucharest and the Institute of Mathematics of the Romanian Academy for the scientific discussions we had with them over the years.

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