1

Monoidal and Braided Categories

In this chapter we introduce the basic categorical language that will be used throughout this book. We define the concepts of monoidal and braided monoidal category and prove that any monoidal category is monoidally equivalent to a strict one.

1.1 Monoidal Categories

Recall that a category $\mathscr{C}$ consists of the following:

- a collection $\mathrm{Ob}(\mathscr{C})$, whose elements are called the objects of $\mathscr{C}$; if X is an object of $\mathscr{C}$, we write either $X \in \mathrm{Ob}(\mathscr{C})$ or simply $X \in \mathscr{C}$;
- for every two objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$, whose elements are denoted by $f : X \to Y$ and called the morphisms from X to Y in $\mathscr{C}$;
- for every object X of $\mathscr{C}$, a specified morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathscr{C}}(X,X)$, called the identity morphism of X;
- for every three objects X, Y, Z of $\mathscr{C}$, a function

$$\circ : \mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}}(X,Z),$$

  called the composition function, that maps a pair (f,g) to $\circ(f,g) := g \circ f$, where $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathscr{C}$.

  These data are subject to the following axioms:

(A) Associativity axiom: for all morphisms $f : X \to Y$, $g : Y \to Z$ and $h : Z \to T$ in $\mathscr{C}$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

(I) Identity axiom: $f \circ \mathrm{Id}_X = f = \mathrm{Id}_Y \circ f$, for every morphism $f : X \to Y$ in $\mathscr{C}$.

A morphism $f : X \to Y$ in $\mathscr{C}$ will also be denoted by $X \xrightarrow{f} Y$. Note that, when there is no danger of confusion, the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$ will often be written as gf instead of $g \circ f$.

A morphism $f : X \to Y$ in $\mathscr{C}$ is called an isomorphism if there exists a morphism $g : Y \to X$ in $\mathscr{C}$, called the inverse of f, such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. Note that the inverse is unique.

If $X \in \mathrm{Ob}(\mathscr{C})$, we denote $\mathrm{End}_{\mathscr{C}}(X) := \mathrm{Hom}_{\mathscr{C}}(X,X)$.

A subcategory $\mathscr{D}$ of a category $\mathscr{C}$ is a collection of some objects and some morphisms of $\mathscr{C}$ in such a way that $\mathscr{D}$ becomes a category with composition and identities from $\mathscr{C}$. Furthermore, we say that $\mathscr{D}$ is a full subcategory when $\mathrm{Hom}_{\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$, for all $X,Y \in \mathrm{Ob}(\mathscr{D})$.

Recall also that a functor F between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of:

- a map $\mathrm{Ob}(F) : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$; we will denote $\mathrm{Ob}(F)(X) = F(X)$, for all $X \in \mathrm{Ob}(\mathscr{C})$;
- a function

$$\mathrm{Hom}_F(X,Y) : \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$$

for any objects X,Y of $\mathscr{C}$; we will denote $\mathrm{Hom}_F(X,Y)(f) = F(f)$, for any morphism $f : X \to Y$ in $\mathscr{C}$.

These data are subject to the following axioms:

(A1) Identities are preserved by F, that is, $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$, for all $X \in \mathscr{C}$.

(A2) Composition is preserved by F, i.e. $F(g \circ f) = F(g) \circ F(f)$, for any morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$.

If $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are two functors then the pointwise composition defines a functor from $\mathscr{C}$ to $\mathscr{E}$. It will be denoted by $G \circ F$, or simply GF when there is no danger of confusion.

If $\mathscr{C}$ is a category, there exists a functor $\mathrm{Id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, called the identity functor on $\mathscr{C}$, which is the identity on both objects and morphisms in $\mathscr{C}$.

A functor $F : \mathscr{C} \to \mathscr{D}$ is called an isomorphism if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that $FG = \mathrm{Id}_{\mathscr{D}}$ and $GF = \mathrm{Id}_{\mathscr{C}}$. Such a functor G, if it exists, is unique and is called the inverse of F. Two categories are isomorphic if there exists an isomorphism between them.

If $F : \mathscr{C} \to \mathscr{D}$ is a functor, we call the full image of F (denoted $\mathrm{Im}(F)$) the full subcategory of $\mathscr{D}$ whose objects are $(F(X))_{X \in \mathrm{Ob}(\mathscr{C})}$.

A natural transformation μ between two functors $F,G : \mathscr{C} \to \mathscr{D}$ consists of a family of morphisms in $\mathscr{D}$, $\mu = (\mu_X : F(X) \to G(X))_{X \in \mathrm{Ob}(\mathscr{C})}$, having the property that $G(f) \circ \mu_X = \mu_Y \circ F(f)$, for any morphism $f : X \to Y$ in $\mathscr{C}$. If, moreover, μ_X is an isomorphism in $\mathscr{D}$, for all $X \in \mathrm{Ob}(\mathscr{C})$, then μ is called a natural isomorphism between F and G.

Finally, if $\mathscr{C}, \mathscr{D}$ are categories then $\mathscr{C} \times \mathscr{D}$ is the category whose

- objects are pairs (X,Y), where X is an object of $\mathscr{C}$ and Y is an object of $\mathscr{D}$;
- morphisms between (X,Y) and (X',Y') are pairs (f,g) consisting of a morphism $f : X \to X'$ in $\mathscr{C}$ and a morphism $g : Y \to Y'$ in $\mathscr{D}$.

The identity morphisms and the composition functions in $\mathscr{C} \times \mathscr{D}$ are canonically defined in terms of those of $\mathscr{C}$ and $\mathscr{D}$. The new category $\mathscr{C} \times \mathscr{D}$ is called the product of $\mathscr{C}$ and $\mathscr{D}$.

Cambridge University Press
978-1-108-42701-2 — Quasi-Hopf Algebras
Daniel Bulacu , Stefaan Caenepeel , Florin Panaite , Freddy Van Oystaeyen
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We can now introduce the concept of monoidal category, which is roughly a category $\mathscr{C}$ endowed with an associative "tensor product" $\otimes : \mathscr{C}\times\mathscr{C}\to\mathscr{C}$, with a unit object $\underline{1}$ and coherence. Rigorously, we have the following:

Definition 1.1 A monoidal category consists of a category $\mathscr{C}$ endowed with a functor $\otimes : \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ (called the tensor product), a distinguished object $\underline{1}\in\mathscr{C}$ (called the unit object of $\mathscr{C}$) and natural isomorphisms (X,Y,Z are arbitrary objects of $\mathscr{C}$)

$$a_{X,Y,Z} : (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z) \ \ \text{(the associativity constraint)},$$
$$l_X : \underline{1}\otimes X\to X \ \ \text{(the left unit constraint)},$$
$$r_X : X\otimes\underline{1}\to X \ \ \text{(the right unit constraint)},$$

satisfying the so-called Pentagon Axiom and Triangle Axiom, namely for any objects $X,Y,Z,T\in\mathscr{C}$ the following diagrams are commutative:

$$\begin{array}{ccccc} ((X\otimes Y)\otimes Z)\otimes T & \xrightarrow{a_{X\otimes Y,Z,T}} & (X\otimes Y)\otimes(Z\otimes T) & \xrightarrow{a_{X,Y,Z\otimes T}} & X\otimes(Y\otimes(Z\otimes T)) \\ \downarrow{\scriptstyle a_{X,Y,Z}\otimes \mathrm{Id}_T} & & & & \uparrow{\scriptstyle \mathrm{Id}_X\otimes a_{Y,Z,T}} \\ (X\otimes(Y\otimes Z))\otimes T & & \xrightarrow[a_{X,Y\otimes Z,T}]{} & & X\otimes((Y\otimes Z)\otimes T), \end{array} \tag{1.1.1}$$

$$\begin{array}{ccc} (X\otimes\underline{1})\otimes Y & \xrightarrow{a_{X,\underline{1},Y}} & X\otimes(\underline{1}\otimes Y) \\ & {\scriptstyle r_X\otimes \mathrm{Id}_Y}\searrow \quad \swarrow{\scriptstyle \mathrm{Id}_X\otimes l_Y} & \\ & X\otimes Y & . \end{array} \tag{1.1.2}$$

The monoidal category $(\mathscr{C},\otimes,\underline{1},a,l,r)$ is called strict if all the natural isomorphisms a,l and r are defined by identity morphisms in $\mathscr{C}$.

Remark 1.2 Let $X\xrightarrow{f}Y\xrightarrow{g}Z$ and $X'\xrightarrow{f'}Y'\xrightarrow{g'}Z'$ be morphisms in $\mathscr{C}$. The fact that $\otimes : \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ is a functor implies the following equality:

$$(g\circ f)\otimes(g'\circ f') = (g\otimes g')\circ(f\otimes f') : X\otimes X'\to Z\otimes Z'.$$

Also, for all objects X,Y of $\mathscr{C}$ we have $\mathrm{Id}_{X\otimes Y} = \mathrm{Id}_X\otimes\mathrm{Id}_Y$.

If $\mathscr{C}$ is a monoidal category and X,Y,Z,T are objects of $\mathscr{C}$, there are two different ways to go from $((X\otimes Y)\otimes Z)\otimes T$ to $X\otimes(Y\otimes(Z\otimes T))$. The Pentagon Axiom says that these two ways coincide. Then it is automatic that all the other consistency problems of this type are solved as well; see Remark 1.35 below.

Proposition 1.3 *Let $(\mathscr{C},\otimes,\underline{1},a,l,r)$ be a monoidal category and consider the switch functor $\tau : \mathscr{C}\times\mathscr{C}\to\mathscr{C}\times\mathscr{C}$, defined by $\tau(X,Y)=(Y,X)$, for any $X,Y\in\mathscr{C}$, and $\tau(f,g)=(g,f)$, for any morphisms $X\xrightarrow{f}X'$ and $Y\xrightarrow{g}Y'$ in $\mathscr{C}$. Then*

$$\overline{\mathscr{C}} := (\mathscr{C},\ \overline{\otimes} := \otimes\circ\tau,\ \overline{a},\ \underline{1},\ \overline{l} := r,\ \overline{r} := l)$$

is a monoidal category, where $\overline{a}_{X,Y,Z} := a^{-1}_{Z,Y,X}$, for all $X,Y,Z\in\mathscr{C}$.

In what follows $\overline{\mathscr{C}}$ will be called the reverse monoidal category associated to $\mathscr{C}$.

CAMBRIDGE
Cambridge University Press
978-1-108-42701-2 — Quasi-Hopf Algebras
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Proof All the axioms for $\overline{\mathscr{C}}$ to be a monoidal category follow from those of $\mathscr{C}$ and the fact that $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$, for any isomorphisms $X\xrightarrow{f}Y\xrightarrow{g}Z$ in $\mathscr{C}$. For example, the Pentagon Axiom for $\overline{\mathscr{C}}$ reduces to the commutativity of the diagram

$$\begin{array}{ccccc} T\otimes(Z\otimes(Y\otimes X)) & \xrightarrow{a^{-1}_{T,Z,Y\otimes X}} & (T\otimes Z)\otimes(Y\otimes X) & \xrightarrow{a^{-1}_{T\otimes Z,Y,X}} & ((T\otimes Z)\otimes Y)\otimes X \\ \downarrow{\scriptstyle \mathrm{Id}_T\otimes a^{-1}_{Z,Y,X}} & & & & \uparrow{\scriptstyle a^{-1}_{T,Z,Y}\otimes \mathrm{Id}_X} \\ T\otimes((Z\otimes Y)\otimes X) & & \xrightarrow[a^{-1}_{T,Z\otimes Y,X}]{} & & (T\otimes(Z\otimes Y))\otimes X, \end{array}$$

which holds because of (1.1.1). Similarly, for $\overline{a}$ as above, $\overline{l}=r$ and $\overline{r}=l$, the Triangle Axiom is satisfied because of (1.1.2). □

Remark 1.4 Apart from $\overline{\mathscr{C}}$, to a monoidal category $\mathscr{C}$ we can associate a new one that will be denoted by $\mathscr{C}^{\mathrm{opp}}$ and called the opposite category associated to $\mathscr{C}$. As a category, $\mathscr{C}^{\mathrm{opp}}$ has the same objects as $\mathscr{C}$ and $\mathrm{Hom}_{\mathscr{C}^{\mathrm{opp}}}(X,Y)=\mathrm{Hom}_{\mathscr{C}}(Y,X)$, for any objects X,Y of $\mathscr{C}$. If $f\in\mathrm{Hom}_{\mathscr{C}^{\mathrm{opp}}}(X,Y)$ and $g\in\mathrm{Hom}_{\mathscr{C}^{\mathrm{opp}}}(Y,Z)$ then the composition $\circ_{\mathrm{opp}}$ between g and f in $\mathscr{C}^{\mathrm{opp}}$ is $g\circ_{\mathrm{opp}}f=f\circ g$, the latest composition being in $\mathscr{C}$.

If $\mathscr{C}$ is monoidal then so is $\mathscr{C}^{\mathrm{opp}}$, with the monoidal structure induced by that of $\mathscr{C}$, namely $\mathscr{C}^{\mathrm{opp}}=(\mathscr{C}^{\mathrm{opp}},\otimes,\underline{1},a^{-1},l^{-1},r^{-1})$.

The Triangle Axiom in Definition 1.1 gives the compatibility between the left and right unit constraints. There also exist other compatibilities of this type:

Proposition 1.5 *Let $(\mathscr{C},\otimes,\underline{1},a,l,r)$ be a monoidal category. Then the diagrams*

$$\begin{array}{ccc} (X\otimes Y)\otimes\underline{1} & \xrightarrow{a_{X,Y,\underline{1}}} & X\otimes(Y\otimes\underline{1}) \\ & {\scriptstyle r_{X\otimes Y}}\searrow \quad \swarrow{\scriptstyle \mathrm{Id}_X\otimes r_Y} & \\ & X\otimes Y & \end{array}$$

and

$$\begin{array}{ccc} (\underline{1}\otimes X)\otimes Y & \xrightarrow{a_{\underline{1},X,Y}} & \underline{1}\otimes(X\otimes Y) \\ & {\scriptstyle l_X\otimes \mathrm{Id}_Y}\searrow \quad \swarrow{\scriptstyle l_{X\otimes Y}} & \\ & X\otimes Y & \end{array}$$

are commutative, for any objects $X,Y\in\mathscr{C}$. Moreover, we have that $l_{\underline{1}}=r_{\underline{1}}$.

Proof Since a is natural, the following diagrams are commutative:

$$\begin{array}{ccc} (X\otimes(Y\otimes\underline{1}))\otimes T & \xrightarrow{a_{X,Y\otimes\underline{1},T}} & X\otimes((Y\otimes\underline{1})\otimes T) \\ \downarrow{\scriptstyle (\mathrm{Id}_X\otimes r_Y)\otimes\mathrm{Id}_T} & & \downarrow{\scriptstyle \mathrm{Id}_X\otimes(r_Y\otimes\mathrm{Id}_T)} \\ (X\otimes Y)\otimes T & \xrightarrow{a_{X,Y,T}} & X\otimes(Y\otimes T), \end{array} \tag{1.1.3}$$

$$\begin{array}{ccc} (X\otimes Y)\otimes(\underline{1}\otimes T) & \xrightarrow{a_{X,Y,\underline{1}\otimes T}} & X\otimes(Y\otimes(\underline{1}\otimes T)) \\ {\scriptstyle \mathrm{Id}_{X\otimes Y}\otimes l_T}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_X\otimes(\mathrm{Id}_Y\otimes l_T)} \\ (X\otimes Y)\otimes T & \xrightarrow{a_{X,Y,T}} & X\otimes(Y\otimes T), \end{array} \tag{1.1.4}$$

for all $X,Y,T\in\mathscr{C}$. Then we have:

$$\begin{aligned} & a_{X,Y,T}((\mathrm{Id}_X\otimes r_Y)\otimes \mathrm{Id}_T)(a_{X,Y,\underline{1}}\otimes \mathrm{Id}_T) \\ & \overset{(1.1.3)}{=} (\mathrm{Id}_X\otimes(r_Y\otimes \mathrm{Id}_T))a_{X,Y\otimes\underline{1},T}(a_{X,Y,\underline{1}}\otimes \mathrm{Id}_T) \\ & \overset{(1.1.2)}{=} (\mathrm{Id}_X\otimes(\mathrm{Id}_Y\otimes l_T))(\mathrm{Id}_X\otimes a_{Y,\underline{1},T})a_{X,Y\otimes\underline{1},T}(a_{X,Y,\underline{1}}\otimes \mathrm{Id}_T) \\ & \overset{(1.1.1)}{=} (\mathrm{Id}_X\otimes(\mathrm{Id}_Y\otimes l_T))a_{X,Y,\underline{1}\otimes T}a_{X\otimes Y,\underline{1},T} \\ & \overset{(1.1.4)}{=} a_{X,Y,T}(\mathrm{Id}_{X\otimes Y}\otimes l_T)a_{X\otimes Y,\underline{1},T} \\ & \overset{(1.1.2)}{=} a_{X,Y,T}(r_{X\otimes Y}\otimes \mathrm{Id}_T). \end{aligned}$$

Using that $a_{X,Y,T}$ is an isomorphism we get, for all $X,Y,T\in\mathscr{C}$,

$$(\mathrm{Id}_X\otimes r_Y)a_{X,Y,\underline{1}}\otimes \mathrm{Id}_T = r_{X\otimes Y}\otimes \mathrm{Id}_T. \tag{1.1.5}$$

Now, by the naturality of r the diagrams

$$\begin{array}{ccc} ((X\otimes Y)\otimes\underline{1})\otimes\underline{1} & \xrightarrow{r_{(X\otimes Y)\otimes\underline{1}}} & (X\otimes Y)\otimes\underline{1} \\ {\scriptstyle (\mathrm{Id}_X\otimes r_Y)a_{X,Y,\underline{1}}\otimes \mathrm{Id}_{\underline{1}}}\downdownarrows{\scriptstyle r_{X\otimes Y}\otimes \mathrm{Id}_{\underline{1}}} & & {\scriptstyle (\mathrm{Id}_X\otimes r_Y)a_{X,Y,\underline{1}}}\downdownarrows{\scriptstyle r_{X\otimes Y}} \\ (X\otimes Y)\otimes\underline{1} & \xrightarrow[r_{X\otimes Y}]{} & X\otimes Y \end{array}$$

are commutative, so by (1.1.5) (with $T=\underline{1}$) we obtain that

$$(\mathrm{Id}_X\otimes r_Y)a_{X,Y,\underline{1}}r_{(X\otimes Y)\otimes\underline{1}} = r_{X\otimes Y}r_{(X\otimes Y)\otimes\underline{1}},$$

and therefore the first triangle in the proposition is commutative because $r_{(X\otimes Y)\otimes\underline{1}}$ is an isomorphism.

If we express the commutativity of the first triangle for $\overline{\mathscr{C}}$, the reverse monoidal category associated to $\mathscr{C}$ as in Proposition 1.3, we obtain the commutativity of the second triangle in the proposition.

So it remains to prove $l_{\underline{1}} = r_{\underline{1}}$. For this, note that the naturality of r implies that

$$\begin{array}{ccc} (X\otimes\underline{1})\otimes\underline{1} & \xrightarrow{r_{X\otimes\underline{1}}} & X\otimes\underline{1} \\ {\scriptstyle r_X\otimes \mathrm{Id}_{\underline{1}}}\downarrow & & \downarrow{\scriptstyle r_X} \\ X\otimes\underline{1} & \xrightarrow[r_X]{} & X \end{array}$$

is commutative, for any $X\in\mathscr{C}$. Since r_X is an isomorphism we deduce that

$$r_{X\otimes\underline{1}} = r_X\otimes \mathrm{Id}_{\underline{1}}. \tag{1.1.6}$$

Note that, by applying equation (1.1.6) in $\overline{\mathscr{C}}$, we obtain in $\mathscr{C}$ the relation

$$l_{\underline{1}\otimes X} = \mathrm{Id}_{\underline{1}} \otimes l_X. \tag{1.1.7}$$

Now, by (1.1.2) we have $r_{\underline{1}\otimes\underline{1}} = r_{\underline{1}} \otimes \mathrm{Id}_{\underline{1}} = (\mathrm{Id}_{\underline{1}} \otimes l_{\underline{1}})a_{\underline{1},\underline{1},\underline{1}}$, and by the commutativity of the first triangle in the proposition we get $r_{\underline{1}\otimes\underline{1}} = (\mathrm{Id}_{\underline{1}} \otimes r_{\underline{1}})a_{\underline{1},\underline{1},\underline{1}}$. Since $a_{\underline{1},\underline{1},\underline{1}}$ is an isomorphism we obtain $\mathrm{Id}_{\underline{1}} \otimes l_{\underline{1}} = \mathrm{Id}_{\underline{1}} \otimes r_{\underline{1}}$.

By the naturality of l the diagrams

$$\begin{array}{ccc}
\underline{1}\otimes(\underline{1}\otimes\underline{1}) & \xrightarrow{\ l_{\underline{1}\otimes\underline{1}}\ } & \underline{1}\otimes\underline{1} \\
{\scriptstyle \mathrm{Id}_{\underline{1}}\otimes l_{\underline{1}}}\downarrow\downarrow{\scriptstyle \mathrm{Id}_{\underline{1}}\otimes r_{\underline{1}}} & & {\scriptstyle l_{\underline{1}}}\downarrow\downarrow{\scriptstyle r_{\underline{1}}} \\
\underline{1}\otimes\underline{1} & \xrightarrow[\ l_{\underline{1}}\ ]{} & \underline{1}
\end{array}$$

are commutative. Using that $l_{\underline{1}\otimes\underline{1}}$ is an isomorphism and $\mathrm{Id}_{\underline{1}} \otimes l_{\underline{1}} = \mathrm{Id}_{\underline{1}} \otimes r_{\underline{1}}$ we get $l_{\underline{1}} = r_{\underline{1}}$, and this finishes the proof. □

Proposition 1.6 *Let $\underline{1}$ be the unit object of a monoidal category $\mathscr{C}$. Then $\mathrm{End}_{\mathscr{C}}(\underline{1})$ is a commutative monoid, and if we identify $\underline{1}\otimes\underline{1}$ with $\underline{1}$ via $l_{\underline{1}} = r_{\underline{1}}$ then the tensor product of two morphisms in $\mathrm{End}_{\mathscr{C}}(\underline{1})$ coincides with their composition.*

Proof It can be easily checked that the composition endows $\mathrm{End}_{\mathscr{C}}(\underline{1})$ with a monoid structure, the unit element being $\mathrm{Id}_{\underline{1}}$. Thus, we only need to show that

$$f\otimes g = r_{\underline{1}}^{-1}\circ(f\circ g)\circ r_{\underline{1}} = r_{\underline{1}}^{-1}\circ(g\circ f)\circ r_{\underline{1}},$$

for all $f, g \in \mathrm{End}_{\mathscr{C}}(\underline{1})$. To this end, note that the naturality of l and r imply the commutativity of the following diagrams:

$$\begin{array}{ccc}
\underline{1}\otimes\underline{1} & \xrightarrow{\ l_{\underline{1}}\ } & \underline{1} \\
{\scriptstyle \mathrm{Id}_{\underline{1}}\otimes g}\downarrow & & \downarrow{\scriptstyle g} \\
\underline{1}\otimes\underline{1} & \xrightarrow{\ l_{\underline{1}}\ } & \underline{1}
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
\underline{1}\otimes\underline{1} & \xrightarrow{\ r_{\underline{1}}\ } & \underline{1} \\
{\scriptstyle f\otimes \mathrm{Id}_{\underline{1}}}\downarrow & & \downarrow{\scriptstyle f} \\
\underline{1}\otimes\underline{1} & \xrightarrow{\ r_{\underline{1}}\ } & \underline{1}\,.
\end{array}$$

Now, since $l_{\underline{1}}$ and $r_{\underline{1}}$ are isomorphisms we obtain $\mathrm{Id}_{\underline{1}}\otimes g = l_{\underline{1}}^{-1}\circ g\circ l_{\underline{1}}$ and $f\otimes \mathrm{Id}_{\underline{1}} = r_{\underline{1}}^{-1}\circ f\circ r_{\underline{1}}$. Since $l_{\underline{1}} = r_{\underline{1}}$ it follows that

$$f\otimes g = (f\otimes \mathrm{Id}_{\underline{1}})\circ(\mathrm{Id}_{\underline{1}}\otimes g) = r_{\underline{1}}^{-1}\circ(f\circ g)\circ r_{\underline{1}},$$
$$f\otimes g = (\mathrm{Id}_{\underline{1}}\otimes g)\circ(f\otimes \mathrm{Id}_{\underline{1}}) = r_{\underline{1}}^{-1}\circ(g\circ f)\circ r_{\underline{1}}.$$

Thus, we proved the equalities $f\otimes g = r_{\underline{1}}^{-1}\circ(f\circ g)\circ r_{\underline{1}} = r_{\underline{1}}^{-1}\circ(g\circ f)\circ r_{\underline{1}}$. Note that by interchanging f and g in the above relation we also obtain $f\otimes g = g\otimes f$, for all $f, g \in \mathrm{End}_{\mathscr{C}}(\underline{1})$. □

1.2 Examples of Monoidal Categories

1.2.1 The Category of Sets

We denote the category of sets by $\underline{\mathrm{Set}}$, and by $\{*\}$ a fixed singleton, that is, a fixed set with one element. Furthermore, by $\times$ we denote the direct product of sets, that is, for any sets X and Y, $X \times Y$ is the set of ordered pairs (x,y) with $x \in X$ and $y \in Y$, and by $f \times f'$ the direct product of two functions $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$, that is, $f \times f' : X \times X' \to Y \times Y'$ is defined by $f \times f'(x,x') = (f(x), f'(x'))$, for all $x \in X$ and $x' \in X'$. If follows that $\times$ defines a functor from $\underline{\mathrm{Set}} \times \underline{\mathrm{Set}}$ to $\underline{\mathrm{Set}}$.

For any sets X, Y and Z, we have canonical isomorphisms, defined for all $x \in X$, $y \in Y$ and $z \in Z$, by

$$a_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z), \;\; a_{X,Y,Z}((x,y),z) = (x,(y,z)),$$
$$l_X : \{*\} \times X \to X, \;\; l_X(*,x) = x,$$
$$r_X : X \times \{*\} \to X, \;\; r_X(x,*) = x.$$

The proof of the next result is straightforward, so it is left to the reader.

Proposition 1.7 *With notation as above,* $(\underline{\mathrm{Set}}, \times, \{*\}, a, l, r)$ *is a monoidal category.*

1.2.2 The Category of Vector Spaces

One of the most important examples of a monoidal category for what follows is the category ${}_k\mathscr{M}$ of vector spaces over a base field k. The tensor product in ${}_k\mathscr{M}$ is the usual tensor product of vector spaces, the unit object $\underline{1}$ is the field k itself, and the associativity and unit constraints are the natural isomorphisms (for all $X, Y, Z \in {}_k\mathscr{M}$)

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \;\; a_{X,Y,Z}((x \otimes y) \otimes z) = x \otimes (y \otimes z),$$
$$l_X : k \otimes X \to X, \;\; l_X(\lambda \otimes x) = \lambda x,$$
$$r_X : X \otimes k \to X, \;\; r_X(x \otimes \lambda) = \lambda x,$$

for all $\lambda \in k$, $x \in X$, $y \in Y$ and $z \in Z$. The above statement remains valid if we consider k a commutative ring and take ${}_k\mathscr{M}$ equal to the category of modules over k.

1.2.3 The Category of Bimodules

We present now the noncommutative version of Subsection 1.2.2.

Let k be a field (or, more generally, a commutative ring) and R a k-algebra. Denote by ${}_R\mathscr{M}_R$ the category of R-bimodules and R-bimodule maps. Then ${}_R\mathscr{M}_R$ is monoidal with the following structure:

- The tensor product functor is $\otimes_R : {}_R\mathscr{M}_R \times {}_R\mathscr{M}_R \to {}_R\mathscr{M}_R$ defined as follows. On objects, we have $\otimes_R(M,N) := M \otimes_R N$, the tensor product over R between M and N. It becomes an R-bimodule in the canonical way: $r \cdot (m \otimes_R n) \cdot r' = rm \otimes_R nr'$, for all $r, r' \in R$, $m \in M$ and $n \in N$. If $f : M \to N$, $g : P \to Q$ are morphisms in ${}_R\mathscr{M}_R$ then the map $(f \otimes_R g)(m \otimes_R p) = f(m) \otimes_R g(p)$, for all $m \in M$ and $p \in P$, is a morphism in ${}_R\mathscr{M}_R$.

- The unit is R, considered as an R-bimodule via its multiplication.
- The associativity and unit constraints are defined as follows:

$$a_{X,Y,Z} : (X \otimes_R Y) \otimes_R Z \to X \otimes_R (Y \otimes_R Z),\ \ a_{X,Y,Z}((x \otimes_R y) \otimes_R z) = x \otimes_R (y \otimes_R z),$$
$$l_X : R \otimes_R X \to X,\ \ l_X(r \otimes_R x) = rx,$$
$$r_X : X \otimes_R R \to X,\ \ r_X(x \otimes_R r) = xr,$$

for all $r \in R$, $x \in X$, $y \in Y$ and $z \in Z$.

We leave it to the reader to check that this defines a monoidal structure on ${}_R\mathscr{M}_R$. Note that if $R = k$ then ${}_R\mathscr{M}_R$ coincides with ${}_k\mathscr{M}$ as a monoidal category.

1.2.4 The Category of G-graded Vector Spaces

Throughout this subsection G is a group written multiplicatively and with neutral element e, k is a field and $k^* = k \backslash \{0\}$.

Definition 1.8 A G-graded vector space over k is a k-vector space V which decomposes into a direct sum of the form $V = \bigoplus_{g\in G} V_g$, where each V_g is a k-vector space. For a given $g \in G$ the elements of V_g are called homogeneous elements of degree g. If $v \in V$ is a homogeneous element then we denote the degree of v by $|\, v \,| \in G$.

Let $W = \bigoplus_{g\in G} W_g$ be another G-graded vector space. Then a k-linear map $f : V \to W$ is called a G-graded morphism if it preserves the degree of homogeneous elements, that is, $f(V_g) \subseteq W_g$, for all $g \in G$.

Vect^G denotes the category of G-graded vector spaces and G-graded morphisms.

If $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ are G-graded vector spaces then $V \otimes W$ is also a G-graded vector space with the grading defined by

$$(V \otimes W)_g := \bigoplus_{\sigma\tau = g} V_\sigma \otimes W_\tau, \tag{1.2.1}$$

for all $g \in G$. Indeed, it is an elementary fact that in ${}_k\mathscr{M}$ the tensor product commutes with arbitrary direct sums. Hence

$$\bigoplus_{g\in G} (V \otimes W)_g = \bigoplus_{g\in G} \left(\bigoplus_{\sigma\tau = g} V_\sigma \otimes W_\tau \right) = \left(\bigoplus_{g\in G} V_g \right) \otimes \left(\bigoplus_{g'\in G} W_{g'} \right) = V \otimes W,$$

as required. Furthermore, if $f : V \to V'$ and $g : W \to W'$ are morphisms in Vect^G then $f \otimes g$ becomes a morphism in Vect^G. Thus, the tensor product $\otimes$ of the category of k-vector spaces induces a tensor product on Vect^G.

Also, k can be viewed as a G-graded vector space via the trivial grading, that is, $k_e = k$ and $k_g = 0$, for all $G \ni g \neq e$. In this way the left and right unit constraints l and r of ${}_k\mathscr{M}$ become graded morphisms, that is, morphisms in Vect^G.

Our next aim is to describe the monoidal structures of Vect^G, somehow induced by the monoidal structure of ${}_k\mathscr{M}$. To this end we first need some group cohomology, with a particular emphasis on the third cohomology group of a group G with coefficients in k^*, the group of units of a field k, viewed trivially as a $\mathbb{Z}[G]$-module. Here $\mathbb{Z}$ is the ring of integers and $\mathbb{Z}[G]$ is the group algebra associated to G over the commutative ring $\mathbb{Z}$. More generally, for G a (multiplicative) group with neutral element e and R a commutative ring we denote by $R[G]$ the free R-module with basis $\{g \mid g \in G\}$, so any element of $R[G]$ has the form $\sum_{g\in G} \alpha_g g$ with $(\alpha_g)_{g\in G}$ a family of elements of R having only a finite number of non-zero elements. Then $R[G]$ with multiplication defined by $(\alpha_h h)(\beta_g g) = \alpha_h \beta_g hg$, extended by linearity, and unit e, is called the group algebra associated to G over R. It is easy to see that $R[G]$ is a unital associative R-algebra, and that $R[G]$ is a G-graded vector space with grading defined by $R[G]_g = Rg$, for all $g \in G$.

Coming back to the survey on group cohomology, let $K^n(G,k^*)$ be the set of maps from G^n to k^*. Then one can easily see that $K^n(G,k^*)$ is a group under pointwise multiplication. There exist maps $\Delta_n :\ K^n(G,k^*) \to K^{n+1}(G,k^*)$, which for $n \in \{2,3\}$ are given by the formulas

$$\Delta_2(g)(x,y,z) = g(y,z)g(xy,z)^{-1}g(x,yz)g(x,y)^{-1},$$
$$\Delta_3(f)(x,y,z,t) = f(y,z,t)f(xy,z,t)^{-1}f(x,yz,t)f(x,y,zt)^{-1}f(x,y,z).$$

It is known that $B^n(G,k^*) := \mathrm{Im}\Delta_{n-1} \subseteq Z^n(G,k^*) := \mathrm{Ker}(\Delta_n)$. The nth cohomology group is defined as $H^n(G,k^*) = Z^n(G,k^*)/B^n(G,k^*)$, and two elements of $H^n(G,k^*)$ are called cohomologous if they lie in the same equivalence class.

The elements of $Z^3(G,k^*)$ are called 3-cocycles, and the elements of $B^3(G,k^*)$ are called 3-coboundaries. We have the following.

Definition 1.9 A 3-cocycle on G with coefficients in k^* is a map $\phi : G\times G\times G \to k^*$ such that

$$\phi(y,z,t)\phi(x,yz,t)\phi(x,y,z) = \phi(x,y,zt)\phi(xy,z,t), \tag{1.2.2}$$

for all $x,y,z,t \in G$. A 3-cocycle ϕ is called normalized if $\phi(x,e,y) = 1$, for all $x,y \in G$.

Remarks 1.10 (1) If ϕ is a normalized 3-cocycle, then $\phi(e,y,z) = \phi(x,y,e) = 1$, for all $x,y,z \in G$.

Indeed, by taking $z = e$ in (1.2.2), we find that $\phi(x,y,e) = 1$. By taking $y = e$, we find that $\phi(e,z,t) = 1$.

(2) A coboundary $\Delta_2(g)$ is normalized if and only if $g(e,x) = g(z,e)$, for all $x,z \in G$.

As we shall see, $H^3(G,k^*)$ is completely determined by the normalized 3-cocycles.

Lemma 1.11 *Every 3-cocycle ϕ is cohomologous to a normalized 3-cocycle.*

Proof By taking $y = z = e$ in (1.2.2) we find $\phi(x,e,t) = \phi(e,e,t)\phi(x,e,e)$. In particular, by taking $x = t = e$, it follows that $\phi(e,e,e) = 1$. Then we consider the map

$f:\ G\times G\to k^*$, $f(x,y)=\phi(e,e,y)^{-1}\phi(x,e,e)$, and compute:

$$\begin{aligned}\Delta_2(f)(x,e,y)&=f(e,y)f(x,y)^{-1}f(x,y)f(x,e)^{-1}\\&=\phi(e,e,y)^{-1}\phi(e,e,e)\phi(e,e,e)\phi(x,e,e)^{-1}=\phi(x,e,y)^{-1}.\end{aligned}$$

It then follows that $\phi\Delta_2(f)$ is normalized. □

Let $B^3_n(G,k^*)$ and $Z^3_n(G,k^*)$ be the subgroups of $B^3(G,k^*)$ and $Z^3(G,k^*)$ consisting of normalized elements. We have a well-defined group morphism

$$Z^3_n(G,k^*)/B^3_n(G,k^*)\ni\hat{\phi}\mapsto\overline{\phi}\in Z^3(G,k^*)/B^3(G,k^*)$$

which is surjective by Lemma 1.11. One can see that it is also injective, and therefore

$$H^3(G,k^*)=Z^3_n(G,k^*)/B^3_n(G,k^*).$$

Example 1.12 If k is a field of characteristic different from 2 and C_2 is the cyclic group of order 2 then $H^3(C_2,k^*)=C_2$. If $\mathrm{char}(k)=2$, then $H^3(C_2,k^*)=\{e\}$.

Proof Write $C_2=\{1,\sigma\}$. A straightforward computation shows that all normalized coboundaries are trivial. If ϕ is a normalized 3-cocycle, then the only value of $\phi(x,y,z)$ that is possibly different from 1 is $\phi(\sigma,\sigma,\sigma)$. By substituting $x=y=z=t=\sigma$ in (1.2.2), we find that $\phi(\sigma,\sigma,\sigma)=\pm1$. If $\phi(\sigma,\sigma,\sigma)=1$, then ϕ is trivial. The only possibly non-trivial normalized 3-cocycle is given by $\phi(\sigma,\sigma,\sigma)=-1$.

Consequently, if $\mathrm{char}(k)=2$ then any normalized 3-cocycle is trivial, and so $H^3(C_2,k^*)=\{e\}$. □

One can now provide the connection between $H^3(G,k^*)$ and some monoidal structures on Vect^G.

Proposition 1.13 *Let G be a group, k a field and Vect^G the category of G-graded k-vector spaces. There is a bijective correspondence between the monoidal structures on Vect^G of the form $(\mathrm{Vect}^G,\otimes,a,k,l,r)$ and the set of normalized 3-cocycles on G, where $\otimes$ is defined by (1.2.1) and l,r are the constraints of the monoidal category ${}_k\mathscr{M}$ as defined in Subsection 1.2.2.*

More precisely, any associativity constraint a on Vect^G is completely determined by a normalized 3-cocycle $\phi\in H^3(G,k^*)$, in the sense that, for any $U,V,W\in\mathrm{Vect}^G$ and any homogeneous elements $u\in U$, $v\in V$ and $w\in W$, $a_{U,V,W}$ is the k-linear map

$$a_{U,V,W}((u\otimes v)\otimes w)=\phi(|\,u\,|,|\,v\,|,|\,w\,|)u\otimes(v\otimes w).$$

We denote by Vect^G_ϕ the category Vect^G with monoidal structure determined by ϕ.

Proof If ϕ is a normalized 3-cocycle on G then, clearly, the morphism $a_{U,V,W}$ defined above preserves the degree of homogeneous elements, so it is a morphism in Vect^G. The Pentagon Axiom (1.1.1) follows now from (1.2.2), while the Triangle Axiom in (1.1.2) follows because ϕ is normalized. The details are straightforward, so they are left to the reader.