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978-1-108-42591-9 — The Block Theory of Finite Group Algebras
Markus Linckelmann
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Introduction

Group representation theory investigates the structural connections between groups and mathematical objects admitting them as automorphism groups. Its most basic instance is the action of a group G on a set M, which is equivalent to a group homomorphism from G to the symmetric group S_M of all permutations of the set M, thus representing G as an automorphism group of the set M. Classical representation theory, developed during the last decade of the 19th century by Frobenius and Schur, investigates the representations of a finite group G as linear automorphism groups of complex vector spaces, or equivalently, modules over the complex group algebra $\mathbb{C}G$. Modular representation theory, initiated by Brauer in the 1930s, considers finite group actions on vector spaces over fields with positive characteristic, and more generally, on modules over complete discrete valuation rings as a link between different characteristics. Integral representation theory considers representations of groups over rings of algebraic integers, with applications in number theory. Topologists have extensively studied the automorphism groups of classifying spaces of groups in connection with K-theory and transformation groups. Methods from homotopy theory and homological algebra have shaped the area significantly.

Within modular representation theory, viewed as the theory of module categories of finite group algebras over complete discrete valuation rings, the starting point of block theory is the decomposition of finite group algebras into indecomposable direct algebra factors, called *block algebras*. The block algebras of a finite group algebra are investigated individually, bearing in mind that the module category of an algebra is the direct sum of the module categories of its blocks. Block theory seeks to gain insight into which way the structure theory of finite groups and the representation theory of block algebras inform each other.

Few algebras are expected to arise as block algebras of finite groups. Narrowing down the pool of possible block algebras with essentially representation

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theoretic methods has been very successful for blocks of finite and tame representation type, but remains a major challenge beyond those cases. Here is a sample list of properties that have to be satisfied by an algebra B which arises as a block algebra of a finite group algebra over a complete discrete valuation ring $\mathcal{O}$ with residue field k of prime characteristic p and field of fractions K of characteristic 0.

- B is symmetric.
- $K \otimes_{\mathcal{O}} B$ is semisimple.
- The canonical map $Z(B) \to Z(k \otimes_{\mathcal{O}} B)$ is surjective.
- B is separably equivalent to $\mathcal{O}P$ for some finite p-group P.
- The Cartan matrix of $k \otimes_{\mathcal{O}} B$ is positive definite, its determinant is a power of p, and its largest elementary divisor is the smallest power of p that annihilates all homomorphism spaces in the $\mathcal{O}$-stable category $\underline{\mathrm{mod}}(B)$.
- The decomposition map from the Grothendieck group of finitely generated $K \otimes_{\mathcal{O}} B$-modules to the Grothendieck group of finitely generated $k \otimes_{\mathcal{O}} B$-modules is surjective.
- B is defined over a finite extension of the p-adic integers $\mathbb{Z}_p$, and $Z(B)$ is defined over $\mathbb{Z}_{(p)}$.
- $k \otimes_{\mathcal{O}} B$ is defined over a finite field $\mathbb{F}_q$, where q is a power of p, and $Z(k \otimes_{\mathcal{O}} B)$ is defined over $\mathbb{F}_p$.

The dominant feature of block theory is the dichotomy of invariants associated with block algebras. Block algebras of finite groups have all the usual 'global' invariants associated with algebras – module categories, derived and stable categories, cohomological invariants including Hochschild cohomology, and numerical invariants such as the numbers of ordinary and modular irreducible characters. Due to their provenance from finite groups, block algebras have further 'local' invariants that cannot be, in general, associated with arbitrary algebras. The prominent conjectures that drive block theory revolve around the interplay between 'global' and 'local' invariants. Source algebras of blocks capture invariants from both worlds, and in an ideal scenario, the above mentioned conjectures would be obtained as a consequence of a classification of the source algebras of blocks with a fixed defect group. In this generality, this has been achieved in two cases, namely for blocks with cyclic and Klein four defect groups. The local structure of a block algebra B includes the following invariants.

- A defect group P of B.
- A fusion system $\mathcal{F}$ of B on P.

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- A class $\alpha \in H^2(\mathcal{F}^c; k^\times)$ such that α restricts on $\mathrm{Aut}_{\mathcal{F}}(Q)$ to the Külshammer–Puig class α_Q, for any Q belonging to the category $\mathcal{F}^c$ of $\mathcal{F}$-centric subgroups in P.
- The number of weights of $(\mathcal{F}, \alpha)$.

A sample list for the global structure of B includes the following invariants of B as an algebra, as well as their relationship with the local invariants.

- The numbers $|\mathrm{Irr}_K(B)|$ and $|\mathrm{IBr}_k(B)|$ of isomorphism classes of simple $K \otimes_{\mathcal{O}} B$-modules and $k \otimes_{\mathcal{O}} B$-modules, respectively, with their heights.
- The $\mathcal{O}$-stable module category $\underline{\mathrm{mod}}(B)$ and its dimension as a triangulated category.
- The bounded derived category $D^b(B)$ and its dimension as a triangulated category.
- The module category $\mathrm{mod}(B)$ as an abelian category, structure and Loewy lengths of projective indecomposable B-modules.
- The generalised decomposition matrix of B.

All of the above local and global invariants can be calculated, at least in principle, from the source algebras of B, and hence methods to determine source algebras are a major theme in this book.

Volume I introduces the broader context and many of the methods that are fundamental to modular group representation theory. Chapter 1 provides background on algebras and introduces some of the main players in this book – group algebras, twisted group algebras as well as category algebras, for the sake of giving a broader picture. Chapter 2 switches the focus from algebras to module categories and functors. Chapter 3 develops the classical representation theory of finite groups – that is, representations over complex vector spaces – just far enough to prove Burnside's $p^a q^b$-Theorem and describe Brauer's characterisation of characters. Turning to modular representation theory, Chapter 4 handles the general theory of algebras over discrete valuation rings. Chapter 5 combines this material with group actions, leading to Green's theory of vertices and sources, Puig's notion of pointed groups, and further fundamental module theoretic results on special classes of modules, as well as Green's Indecomposability Theorem.

In Volume II, the core theme of this book takes centre stage. Chapter 6 develops in a systematic way block theory, including Brauer's three main theorems, some Clifford Theory, the work of Alperin and Broué on Brauer pairs and Puig's notion of source algebras. Chapter 7 describes modules over finite p-groups, with an emphasis on endopermutation modules. This is followed by another core chapter on local structure, containing in particular a brief

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introduction to fusion systems, connections between characters and local structure, the structure theory of nilpotent blocks and their extensions. Chapter 9 on isometries illustrates the interaction between the concepts introduced up to this point. Applications in subsequent chapters include the structure theory of blocks with cyclic or Klein four defect groups.

Along the way, some of the fundamental conjectures alluded to above will be described. Alperin's weight conjecture predicts that the number of isomorphism classes of simple modules of a block algebra should be determined by its local invariants. Of a more structural nature, Broué's abelian defect group conjecture would offer, if true, some explanation for these numerical coincidences at least in the case of blocks with abelian defect groups. The finiteness conjectures of Donovan, Feit and Puig predict that once a defect group is fixed, there are only 'finitely many blocks' with certain properties. These conjectures are known to hold for blocks of various classes of finite simple groups and their extensions. Complemented by rapidly evolving reduction techniques, this points to the possibility of proving parts of these conjectures by invoking the classification of finite simple groups. A lot more work seems to be needed to provide the understanding that would transform mystery into insight.

The representation theory of finite group algebras draws significantly on methods from areas including ring theory, category theory, and homological algebra. Rather than giving systematic introductions to those areas, we develop background material as we go along, trying not to lose sight of the actual topic.

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