Global Homotopy Theory

Equivariant homotopy theory started from geometrically motivated questions about symmetries of manifolds. Several important equivariant phenomena occur not just for a particular group, but in a uniform way for all groups. Prominent examples include stable homotopy, K-theory or bordism. Global equivariant homotopy theory studies such uniform phenomena, i.e., universal symmetries encoded by simultaneous and compatible actions of all compact Lie groups.

This book introduces graduate students and researchers to global equivariant homotopy theory. The framework is based on the new notion of global equivalences for orthogonal spectra, a much finer notion of equivalence than is traditionally considered. The treatment is largely self-contained and contains many examples, making it suitable as a textbook for an advanced graduate class. At the same time, the book is a comprehensive research monograph with detailed calculations that reveal the intrinsic beauty of global equivariant phenomena.

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Contents

Cambridge University Press
978-1-108-42581-0 — Global Homotopy Theory
Stefan Schwede
Frontmatter
More Information

Preface

Equivariant stable homotopy theory has a long tradition, starting from geometrically motivated questions about symmetries of manifolds. The homotopy-theoretic foundations of the subject were laid by tom Dieck, Segal and May and their students and collaborators in the 1970s, and over the intervening decades equivariant stable homotopy theory has been very useful for solving computational and conceptual problems in algebraic topology, geometric topology and algebraic K-theory. Various important equivariant theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class. Prominent examples of this are equivariant stable homotopy, equivariant K-theory or equivariant bordism. *Global* equivariant homotopy theory studies such uniform phenomena, i.e., the adjective 'global' refers to simultaneous and compatible actions of all compact Lie groups.

This book introduces a new context for global homotopy theory. Various ways of providing a home for global stable homotopy types have previously been explored in [100, Ch. II], [68, Sec. 5], [18] and [19]. We use a different approach: we work with the well-known category of orthogonal spectra, but use a notion of equivalence, the *global equivalence*, which is much finer than what is traditionally considered. The basic underlying observation is that an orthogonal spectrum gives rise to an orthogonal G-spectrum for every compact Lie group G, and the fact that all these individual equivariant objects come from one orthogonal spectrum implicitly encodes strong compatibility conditions as the group G varies. An orthogonal spectrum thus has G-equivariant homotopy groups for every compact Lie group, and a global equivalence is a morphism of orthogonal spectra that induces isomorphisms for all equivariant homotopy groups for all compact Lie groups (based on 'complete G-universes', compare Definition 4.1.3).

The structure of the equivariant homotopy groups of an orthogonal spectrum gives an idea of the information encoded in a global homotopy type in our sense: the equivariant homotopy groups $\pi_k^G(X)$ are contravariantly functorial for continuous group homomorphisms (‘restriction maps’), and they are covariantly functorial for inclusions of closed subgroups (‘transfer maps’). The restriction and transfer maps enjoy various transitivity properties and interact via a double coset formula. This kind of algebraic structure has been studied before under different names, e.g., ‘global Mackey functor’, ‘inflation functor’, ... From a purely algebraic perspective, there are various parameters here that one can vary, namely the class of groups to which a value is assigned and the classes of homomorphisms to which restriction maps or transfer maps are assigned, and lots of variations have been explored. However, the decision to work with orthogonal spectra and equivariant homotopy groups on complete universes dictates a canonical choice: we prove in Theorem 4.2.6 that the algebra of natural operations between the equivariant homotopy groups of orthogonal spectra is freely generated by restriction maps along continuous group homomorphisms and transfer maps along closed subgroup inclusion, subject to explicitly understood relations.

We define the *global stable homotopy category* $\mathcal{GH}$ by localizing the category of orthogonal spectra at the class of global equivalences. Every global equivalence is in particular a non-equivariant stable equivalence, so there is a ‘forgetful’ functor $U\colon \mathcal{GH} \longrightarrow \mathcal{SH}$ on localizations, where $\mathcal{SH}$ denotes the traditional non-equivariant stable homotopy category. By Theorem 4.5.1 this forgetful functor has a left adjoint L and a right adjoint R, both fully faithful, which participate in a recollement of triangulated categories:

$$\mathcal{GH}^+ \begin{smallmatrix} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow[\;i^!\;]{} \end{smallmatrix} \mathcal{GH} \begin{smallmatrix} \xleftarrow{\;L\;} \\ \xrightarrow{\;U\;} \\ \xleftarrow[\;R\;]{} \end{smallmatrix} \mathcal{SH} .$$

Here $\mathcal{GH}^+$ denotes the full subcategory of the global stable homotopy category spanned by the orthogonal spectra that are stably contractible in the traditional, non-equivariant sense.

The global sphere spectrum and suspension spectra are in the image of the left adjoint (Example 4.5.11). Global Borel cohomology theories are the image of the right adjoint (Example 4.5.19). The ‘natural’ global versions of Eilenberg–MacLane spectra (Construction 5.3.8), Thom spectra (Section 6.1), or topological K-theory spectra (Sections 6.3 and 6.4) are not in the image of either of the two adjoints. Periodic global K-theory, however, is right induced from finite cyclic groups, i.e., in the image of the analogous right adjoint from an intermediate global homotopy category $\mathcal{GH}_{cyc}$ based on finite cyclic groups (Example 6.4.27).

Looking at orthogonal spectra through the eyes of global equivalences is like using a prism: the latter breaks up white light into a spectrum of colors, and global equivalences split a traditional, non-equivariant homotopy type into many different global homotopy types. The first example of this phenomenon that we will encounter refines the classifying space of a compact Lie group G. On the one hand, there is the constant orthogonal space with value a non-equivariant model for BG; and there is the *global classifying space* $B_{gl}G$ (see Definition 1.1.27). The global classifying space is analogous to the geometric classifying space of a linear algebraic group in motivic homotopy theory [123, 4.2], and it is the counterpart to the stack of G-principal bundles in the world of stacks.

Another good example is the splitting up of the non-equivariant homotopy type of the classifying space of the infinite orthogonal group. Again there is the constant orthogonal space with value BO, the Grassmannian model **BO** (Example 2.4.1), a different Grassmannian model **bO** (Example 2.4.18), the bar construction model **B°O** (Example 2.4.14), and finally a certain 'cofree' orthogonal space $R(BO)$. The orthogonal space **bO** is also a homotopy colimit, as n goes to infinity, of the global classifying spaces $B_{gl}O(n)$. We discuss these different global forms of BO in some detail in Section 2.4, and the associated Thom spectra in Section 6.1.

In the stable global world, every non-equivariant homotopy type has two extreme global refinements, the 'left induced' (the global analog of a constant orthogonal space, see Example 4.5.10) and the 'right induced' global homotopy type (representing Borel cohomology theories, see Example 4.5.19). Many important stable homotopy types have other natural global forms. The non-equivariant Eilenberg–MacLane spectrum of the integers has a 'free abelian group functor' model (Construction 5.3.8), and another incarnation is the Eilenberg–MacLane spectrum of the constant global functor with value $\mathbb{Z}$ (Remark 4.4.12). These two global refinements of the integral Eilenberg–MacLane spectrum agree on finite groups, but differ for compact Lie groups of positive dimensions.

As already indicated, there is a great variety of orthogonal Thom spectra, in real (or unoriented) flavors as **mO** and **MO**, as complex (or unitary) versions **mU** and **MU**, and there are periodic versions **mOP**, **MOP**, **mUP** and **MUP** of these; we discuss these spectra in Section 6.1. The theories represented by **mO** and **mU** have the closest ties to geometry; for example, the equivariant homotopy groups of **mO** receive Thom–Pontryagin maps from equivariant bordism rings, and these are isomorphisms for products of finite groups and tori (compare Theorem 6.2.33). The theories represented by **MO** are tom Dieck's homotopical equivariant bordism, isomorphic to 'stable equivariant bordism'.

Connective topological K-theory also has two fairly natural global refinements, in addition to the left and right induced ones. The 'orthogonal subspace' model **ku** (Construction 6.3.9) represents connective equivariant

K-theory on the class of finite groups; on the other hand, global connective K-theory $\mathbf{ku}^c$ (Construction 6.4.32) is the global synthesis of equivariant connective K-theory in the sense of Greenlees [66]. The periodic global K-theory spectrum **KU** is introduced in Construction 6.4.9; as the name suggests, **KU** is Bott periodic and represents equivariant K-theory.

The global equivalences are part of a closed model structure (see Theorem 4.3.18), so the methods of homotopical algebra can be used to study the stable global homotopy category. This works more generally relative to a class $\mathcal{F}$ of compact Lie groups, where we define $\mathcal{F}$-equivalences by requiring that $\pi_k^G(f)$ is an isomorphism for all integers and all groups in $\mathcal{F}$. We call a class $\mathcal{F}$ of compact Lie groups a *global family* if it is closed under isomorphisms, subgroups and quotients. For global families we refine the $\mathcal{F}$-equivalences to a stable model structure, the *$\mathcal{F}$-global model structure*, see Theorem 4.3.17. Besides all compact Lie groups, interesting global families are the classes of all finite groups, or all abelian compact Lie groups. The class of trivial groups is also admissible here, but then we just recover the 'traditional' stable category. If the family $\mathcal{F}$ is multiplicative, then the $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra and satisfies the monoid axiom (Proposition 4.3.28). Hence this model structure lifts to modules over an orthogonal ring spectrum and to algebras over an ultra-commutative ring spectrum (Corollary 4.3.29).

Ultra-commutativity A recurring theme throughout this book is a phenomenon that I call *ultra-commutativity*. I use this term in the unstable and stable context for the homotopy theory of strictly commutative objects under global equivalences. An ultra-commutative multiplication has significantly more structure than just a coherently homotopy-commutative product (usually called an E_∞-multiplication). For example, the extra structure gives rise to power operations that can be turned into transfer maps (in additive notation) and norm maps (in multiplicative notation). Another difference is that an unstable global E_∞-structure would give rise to naive deloopings (i.e., by trivial representations). As I hope to discuss elsewhere, a global ultra-commutative multiplication, in contrast, gives rise to 'genuine' deloopings (i.e., by non-trivial representations). As far as the objects are concerned, ultra-commutative monoids and ultra-commutative ring spectra are not at all new and have been much studied before; so one could dismiss the name 'ultra-commutativity' as a mere marketing maneuver. However, the homotopy theory of ultra-commutative monoids and ultra-commutative ring spectra with respect to global equivalences is new and, in the author's opinion, important. And important concepts deserve catchy names.

Global homotopy types as orbifold cohomology theories I would like to briefly mention another reason why one might be interested in global stable homotopy theory. In short, global stable homotopy types represent genuine cohomology theories on stacks, orbifolds and orbispaces. Stacks and orbifolds are concepts from algebraic geometry and geometric topology that allow us to talk about objects that locally look like the quotient of a smooth object by a group action, in a way that remembers information about the isotropy groups of the action. Such 'stacky' objects can behave like smooth objects even if the underlying spaces have singularities. As for spaces, manifolds and schemes, cohomology theories are important invariants also for stacks and orbifolds, and examples such as ordinary cohomology or K-theory lend themselves to generalization. Special cases of orbifolds are 'global quotients', often denoted $M/\!\!/G$, for example for a smooth action of a compact Lie group G on a smooth manifold M. In such examples, the orbifold cohomology of $M/\!\!/G$ is supposed to be the G-equivariant cohomology of M. This suggests a way to *define* orbifold cohomology theories by means of equivariant stable homotopy theory, via suitable G-spectra E_G. However, since the group G is not intrinsic and can vary, one needs equivariant cohomology theories for all groups G, with some compatibility.

Part of the compatibility can be deduced from the fact that the same orbifold can be presented in different ways; for example, if G is a closed subgroup of K, then the global quotients $M/\!\!/G$ and $(M \times_G K)/\!\!/K$ describe the same orbifold. So if the orbifold cohomology theory is represented by equivariant spectra $\{E_G\}_G$ as indicated above, then necessarily $E_G \simeq \operatorname{res}^K_G(E_K)$, i.e., the equivariant homotopy types are consistent under restriction. This is the characteristic feature of *global* equivariant homotopy types, and it suggests that the latter ought to define orbifold cohomology theories.

The approach to global homotopy theory presented in this book in particular provides a way of turning the above outline into rigorous mathematics. There are different formal frameworks for stacks and orbifolds (algebro-geometric, smooth, topological), and these objects can be studied with respect to various notions of 'equivalence'. The approach that most easily feeds into the present context is the notion of *topological stacks* and *orbispaces* as developed by Gepner and Henriques in their paper [61]. Their homotopy theory of topological stacks is rigged up so that the derived mapping spaces out of the classifying stacks for principal G-bundles detect equivalences. In our setup, the global classifying spaces of compact Lie groups (see Definition 1.1.27) play exactly the same role, and this is another hint of a deeper connection. In fact, the global homotopy theory of orthogonal spaces as developed in Chapter 1 is a model for the homotopy theory of orbispaces in the sense of Gepner and Henriques. For a formal comparison of the two models I refer the reader to the author's paper [145]. The comparison proceeds through yet another model, the global homotopy theory of 'spaces with an action of the universal compact Lie group'. Here the universal compact Lie group (which is neither compact nor a Lie group) is the topological monoid $\mathcal{L}$ of linear isometric self-embeddings of $\mathbb{R}^\infty$, and in [145] we establish a global model structure on the category of $\mathcal{L}$-spaces.

If we now accept that one can pass between stacks, orbispaces and orthogonal spaces in a homotopically meaningful way, a consequence is that every global stable homotopy type (i.e., every orthogonal spectrum) gives rise to a cohomology theory on stacks and orbifolds. Indeed, by taking the unreduced suspension spectrum, every unstable global homotopy type is transferred into the triangulated global stable homotopy category $\mathcal{GH}$. In particular, taking morphisms in $\mathcal{GH}$ into an orthogonal spectrum E defines $\mathbb{Z}$-graded E-cohomology groups. The counterpart of a global quotient $M /\!\!/ G$ in the global homotopy theory of orthogonal spaces is the semifree orthogonal space $\mathbf{L}_{G,V} M$ introduced in Construction 1.1.22. By the adjunction relating the global and G-equivariant stable homotopy categories (see Theorem 4.5.24), the morphisms $[\![\Sigma^\infty_+ \mathbf{L}_{G,V} M, E]\!]$ in the global stable homotopy category are in bijection with the G-equivariant E-cohomology groups of M. In other words, when evaluated on a global quotient $M /\!\!/ G$, our recipe for generating an orbifold cohomology theory from a global stable homotopy type precisely returns the G-equivariant cohomology of M, which was the original design criterion.

The procedure sketched so far actually applies to more general objects than our global stable homotopy types: indeed, all that was needed to produce the orbifold cohomology theory was a sufficiently exact functor from the homotopy theory of orbispaces to a triangulated category. If we aim for a stable homotopy theory (as opposed to its triangulated homotopy category), then there is a universal example, namely the stabilization of the homotopy theory of orbispaces, obtained by formally inverting suspension. Our global theory is, however, richer than this 'naive' stabilization. Indeed, there is a forgetful functor from the global stable homotopy category, based on a complete G-universe; the equivariant cohomology theories represented by such objects are usually called 'genuine' (as opposed to 'naive'). Genuine equivariant cohomology theories have much more structure than naive ones; this structure manifests itself in different forms, for example as transfer maps, stability under 'twisted suspension' (i.e., smash product with linear representation spheres), an extension of the $\mathbb{Z}$-graded cohomology groups to an $RO(G)$-graded theory, and an equivariant refinement of additivity (the so called *Wirthmüller isomorphism*). Hence global stable homotopy types in the sense of this book represent *genuine* (as opposed to 'naive') orbifold cohomology theories.

Organization In Chapter 1 we set up unstable global homotopy theory using orthogonal spaces, i.e., continuous functors from the category of finite-dimensional inner product spaces and linear isometric embeddings to spaces. We introduce global equivalences (Definition 1.1.2), discuss global classifying spaces of compact Lie groups (Definition 1.1.27), and set up the global model structures on the category of orthogonal spaces (Theorem 1.2.21). In Section 1.3 we investigate the box product of orthogonal spaces from a global equivariant perspective. Section 1.4 introduces a variant of unstable global homotopy theory based on a *global family*, i.e., a class $\mathcal{F}$ of compact Lie groups with certain closure properties. We discuss the $\mathcal{F}$-global model structure and record that for multiplicative global families, it lifts to the category of modules and algebras (Corollary 1.4.15). In Section 1.5 we discuss the G-equivariant homotopy sets of orthogonal spaces and identify the natural structure between them (restriction maps along continuous group homomorphisms). The study of natural operations on $\pi_0^G(Y)$ is a recurring theme throughout this book; in the later chapters we return to it in the contexts of ultra-commutative monoids, orthogonal spectra and ultra-commutative ring spectra.

Chapter 2 is devoted to ultra-commutative monoids (a.k.a. commutative monoids with respect to the box product, or lax symmetric monoidal functors), which we want to advertise as a rigidified notion of 'global E_∞-space'. In Section 2.1 we establish a global model structure for ultra-commutative monoids (Theorem 2.1.15). Section 2.2 introduces and studies global power monoids, the algebraic structure that an ultra-commutative multiplication gives rise to on the homotopy group Rep-functor $\underline{\pi}_0(R)$. Section 2.3 contains a large collection of examples of ultra-commutative monoids and interesting morphisms between them. In Section 2.4 we discuss and compare different global refinements of the non-equivariant homotopy type BO, the classifying space for the infinite orthogonal group. Section 2.5 discusses 'units' and 'group completions' of ultra-commutative monoids. As an application of this technology we formulate and prove a global, highly structured version of Bott periodicity, see Theorem 2.5.41.

Chapter 3 is a largely self-contained exposition of many basic facts about equivariant stable homotopy theory for a fixed compact Lie group, modeled by orthogonal G-spectra. In Section 3.1 we recall orthogonal G-spectra and equivariant homotopy groups and prove their basic properties, such as the suspension isomorphism and long exact sequences of mapping cones and homotopy fibers, and the additivity of equivariant homotopy groups on sums and products. Section 3.2 discusses the Wirthmüller isomorphism and the closely related transfers. In Section 3.3 we introduce and study geometric fixed-point homotopy groups, an alternative invariant for characterizing equivariant stable equivalences. Section 3.4 contains a proof of the double coset formula for the composite of a transfer followed by the restriction to a closed subgroup. We review Mackey functors for finite groups and show that after inverting the group order, the category of G-Mackey functors splits as a product, indexed by conjugacy classes of subgroups, of module categories over the Weyl groups (Theorem 3.4.22). A topological consequence is that after inverting the group order, equivariant homotopy groups and geometric fixed-point homotopy groups determine each other in a completely algebraic fashion, compare Proposition 3.4.26 and Corollary 3.4.28. Section 3.5 is devoted to multiplicative aspects of equivariant stable homotopy theory.

Chapter 4 sets the stage for stable global homotopy theory, based on the notion of global equivalences for orthogonal spectra (Definition 4.1.3). We discuss semifree orthogonal spectra and identify certain morphisms between semifree orthogonal spectra as global equivalences (Theorem 4.1.29). In Section 4.2 we investigate *global functors*, the natural algebraic structure on the collection of equivariant homotopy groups of a global stable homotopy type. Among other things, we explicitly calculate the algebra of natural operations on equivariant homotopy groups (Theorem 4.2.6). In Section 4.3 we complement the global equivalences of orthogonal spectra by a stable model structure that we call the *global model structure*. Its fibrant objects are the 'global Ω-spectra' (Definition 4.3.8), the natural concept of a 'global infinite loop space' in our setting. Here we work more generally relative to a global family $\mathcal{F}$ and consider the $\mathcal{F}$-equivalences (i.e., equivariant stable equivalences for all compact Lie groups in the family $\mathcal{F}$). We follow the familiar outline: a certain $\mathcal{F}$-level model structure is Bousfield localized to an $\mathcal{F}$-global model structure (see Theorem 4.3.17). In Section 4.4 we develop some basic theory around the global stable homotopy category; since it comes from a stable model structure, this category is naturally triangulated and we show that the suspension spectra of global classifying spaces form a set of compact generators (Theorem 4.4.3). In Section 4.5 we vary the global family: we construct and study left and right adjoints to the forgetful functors associated with a change of global family (Theorem 4.5.1). As an application of Morita theory for stable model categories we show that rationally the global homotopy category for finite groups has an algebraic model, namely the derived category of rational global functors (Theorem 4.5.29).

Chapter 5 focuses on *ultra-commutative ring spectra*, i.e., commutative orthogonal ring spectra under multiplicative global equivalences. Section 5.1 introduces 'global power functors', the algebraic structure on the equivariant homotopy groups of ultra-commutative ring spectra. Roughly speaking, global power functors are global Green functors equipped with additional power operations, satisfying various properties reminiscent of those of the power maps $x \mapsto x^m$ in a commutative ring. The power operations give rise to norm maps ('multiplicative transfers') along finite index inclusions, and in our global context, the norm maps conversely determine the power operations, compare Remark 5.1.7. As we show in Theorem 5.1.11, the 0th equivariant homotopy groups of an ultra-commutative ring spectrum form a global power functor. In Section 5.2 we develop a description of the category of global power functors via the comonad of 'exponential sequences' (Theorem 5.2.13) and discuss localization of global power functors at a multiplicative subset of the underlying ring (Theorem 5.2.18). In Section 5.3 we give various examples of global power functors, such as the Burnside ring global power functor, the global functor represented by an abelian compact Lie group, free global power functors, constant global power functors, and the complex representation ring global functor. In Section 5.4 we establish the global model structure for ultra-commutative ring spectra (Theorem 5.4.3) and show that every global power functor is realized by an ultra-commutative ring spectrum (Theorem 5.4.14).

Chapter 6 is devoted to interesting examples of ultra-commutative ring spectra. Section 6.1 discusses two orthogonal Thom spectra **mO** and **MO**. The spectrum **mO** is globally connective and closely related to equivariant bordism. The global functor $\underline{\pi}_0(\mathbf{mO})$ admits a short and elegant algebraic presentation: it is obtained from the Burnside ring global functor by imposing the single relation $\mathrm{tr}_e^{C_2} = 0$, compare Theorem 6.1.44. The Thom spectrum **MO** was first considered by tom Dieck and it represents 'stable' equivariant bordism; it is periodic for orthogonal representations of compact Lie groups, and admits Thom isomorphisms for equivariant vector bundles. The equivariant homology theory represented by **MO** can be obtained from the one represented by **mO** in an algebraic fashion, by inverting the collection of 'inverse Thom classes', compare Corollary 6.1.35. Section 6.2 recalls the geometrically defined equivariant bordism theories. The Thom–Pontryagin construction maps the unoriented G-equivariant bordism ring $\mathcal{N}_*^G$ to the equivariant homotopy ring $\pi_*^G(\mathbf{mO})$, and that map is an isomorphism when G is a product of a finite group and a torus, see Theorem 6.2.33. We discuss global K-theory in Sections 6.3 and 6.4, which comes in three interesting flavors as connective global K-theory **ku**, global connective K-theory $\mathbf{ku}^c$ and periodic global K-theory **KU** (and in the real versions **ko**, $\mathbf{ko}^c$ and **KO**).

We include three appendices where we collect material that is mostly well-known, but that is either scattered through the literature or where we found the existing expositions too sketchy. Appendix A is a self-contained review of compactly generated spaces, our basic category to work in. Appendix B deals with fundamental properties of equivariant spaces, including the basic model structure in Proposition B.7. We also provide an exposition of the equivariant $\mathbf{\Gamma}$-space machinery, culminating in a version of the Segal–Shimakawa delooping machine. In Appendix C we review the basic definitions, properties and constructions involving categories of enriched functors.

While most of the material in the appendices is well-known, there are a few results I could not find in the literature. These results include the fact that compactly generated spaces are closed under geometric realization (Proposition A.35 (iii)), fixed points commute with geometric realization and latching objects (Proposition B.1 (iv)), and compactly generated spaces are closed under prolongation of $\mathbf{\Gamma}$-spaces (Proposition B.26). Also apparently new are the results that prolongation of G-cofibrant $\mathbf{\Gamma}$-G-spaces to finite G-CW-complexes is homotopically meaningful (Proposition B.48), and that prolongation of G-cofibrant $\mathbf{\Gamma}$-G-spaces to spheres gives rise to G-Ω-spectra (for very special $\mathbf{\Gamma}$-G-spaces, see Theorem B.61) and to positive G-Ω-spectra (for special $\mathbf{\Gamma}$-G-spaces, see Theorem B.65). Here the key ideas all go back to Segal [155] and Shimakawa [157]; however, we formulate our results for the prolongation (i.e., categorical Kan extension), whereas Segal and Shimakawa work with a bar construction (also known as a homotopy coend or homotopy Kan extension) instead. We also give a partial extension of the machinery to compact Lie groups, whereas previous papers on the subject restrict attention to finite groups. As I explain in Remark B.66, there is no hope of obtaining a G-Ω-spectrum by evaluation on spheres for compact Lie groups of positive dimension. However, we do prove in Theorem B.65 that evaluating a G-cofibrant special $\mathbf{\Gamma}$-G-space on spheres yields a 'G°-trivial positive G-Ω-spectrum', where G° is the identity component of G. Our Appendix B substantially overlaps with the paper [115] by May, Merling and Osorno that provides comparisons of prolongation, bar construction and the operadic approach to equivariant deloopings.

Relation to other work The idea of global equivariant homotopy theory is not at all new and has previously been explored in different contexts. For example, in Chapter II of [100], Lewis and May define *coherent families* of equivariant spectra; these consist of collections of equivariant coordinate-free spectra in the sense of Lewis, May and Steinberger, equipped with comparison maps involving change of groups and change of universe functors.

The approach closest to ours is the *global $\mathcal{I}_*$-functors* introduced by Greenlees and May in [68, Sec. 5]. These objects are 'global orthogonal spectra' in that they are indexed on pairs (G, V) consisting of a compact Lie group and a G-representation V. The corresponding objects with commutative multiplication are called *global $\mathcal{I}_*$-functors with smash products* in [68, Sec. 5] and it is for these that Greenlees and May define and study multiplicative norm maps. Clearly, an orthogonal spectrum gives rise to global $\mathcal{I}_*$-functors in the sense of Greenlees and May. In the second chapter of her thesis [18], Bohmann compares the approaches of Lewis–May and Greenlees–May; in the paper [19] she also relates these to orthogonal spectra.

Symmetric spectra in the sense of Hovey, Shipley and Smith [81] is another prominent model for the (non-equivariant) stable homotopy category. Much of what we do here with orthogonal spectra can also be done with symmetric spectra, if one is willing to restrict to finite groups (as opposed to general compact Lie groups). This restriction arises because only finite groups embed into symmetric groups, while every compact Lie group embeds into an orthogonal group. Hausmann [72, 73] has established a global model structure on the category of symmetric spectra, and he showed that the forgetful functor is a right Quillen equivalence from the category of orthogonal spectra with the $\mathcal{F}in$-global model structure to the category of symmetric spectra with the global model structure. While some parts of the symmetric and orthogonal theories are similar, there are serious technical complications arising from the fact that for symmetric spectra the naively defined equivariant homotopy groups are not 'correct', a phenomenon that is already present non-equivariantly.

Prerequisites This book assumes a solid background in algebraic topology and (non-equivariant) homotopy theory, including topics such as singular homology and cohomology, CW-complexes, homotopy groups, mapping spaces, loop spaces, fibrations and fiber bundles, Eilenberg–MacLane spaces, smooth manifolds, Grassmannian and Stiefel manifolds. Two modern references that contain all we need (and much more) are the textbooks by Hatcher [71] and tom Dieck [180]. Some knowledge of non-equivariant stable homotopy theory is helpful to appreciate the equivariant and global features of the structures and examples we discuss; from a strictly logical perspective, however, the non-equivariant theory is a degenerate special case of the global theory for the global family of trivial Lie groups. In particular, by simply ignoring all group actions, the examples presented in this book give models for many interesting and prominent non-equivariant stable homotopy types.

Since actions of compact Lie groups are central to this book, some familiarity with the structure and representation theory of compact Lie groups is obviously helpful, but we give references to the literature whenever we need any non-trivial facts. Many of our objects of study organize themselves into model categories in the sense of Quillen [134], so some background on model categories is necessary to understand the respective sections. The article [48] by Dwyer and Spalinski is a good introduction, and Hovey's book [80] is still the definitive reference. Some acquaintance with unstable equivariant homotopy theory is useful (but not logically necessary). By contrast, we do not assume any prior knowledge of equivariant *stable* homotopy theory, and Chapter 3 is a self-contained introduction based on equivariant orthogonal spectra. The last two sections of Chapter 4 study the global stable homotopy category, and here we freely use the language of triangulated categories. The first chapter of Neeman's book [128] is a possible reference for the necessary background.

Throughout the book we work in the category of compactly generated spaces in the sense of McCord [118], i.e., k-spaces (also called *Kelley spaces*) that satisfy the weak Hausdorff condition, see Definition A.1. Since the various useful properties of compactly generated spaces are scattered through the literature, we include a detailed discussion in Appendix A.

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