1

Unstable global homotopy theory

In this chapter we develop a framework for unstable global homotopy theory via *orthogonal spaces*, i.e., continuous functors from the category of linear isometries **L** to spaces. In Section 1.1 we define global equivalences of orthogonal spaces and establish many basic properties of this class of morphisms. We also introduce global classifying spaces of compact Lie groups, the basic building blocks of global homotopy types. In Section 1.2 we complement the global equivalences by a global model structure on the category of orthogonal spaces. The construction follows a familiar pattern, by Bousfield localization of an auxiliary 'strong level model structure'. Section 1.2 also contains a discussion of cofree orthogonal spaces, i.e., global homotopy types that are 'right induced' from non-equivariant homotopy types. In Section 1.3 we recall the box product of orthogonal spaces, a Day convolution product based on the orthogonal direct sum of inner product spaces. The box product is a symmetric monoidal product, fully homotopical under global equivalences and globally equivalent to the cartesian product. Section 1.4 introduces an important variation of our theme, where we discuss unstable global homotopy theory for a 'global family', i.e., a class of compact Lie groups with certain closure properties. In Section 1.5 we introduce the G-equivariant homotopy set $\pi_0^G(Y)$ of an orthogonal space and identify the natural structure on these sets (restriction maps along continuous group homomorphisms). The study of natural operations on the sets $\pi_0^G(Y)$ is a recurring theme throughout this book, and we will revisit and extend the results on these operations in the later chapters for ultra-commutative monoids, orthogonal spectra and ultra-commutative ring spectra.

Our main reason for working with orthogonal spaces is that they are the direct unstable analog of orthogonal spectra, and in this unstable model for global homotopy theory the passage to the stable theory in Chapter 4 is especially simple. However, there are other models for unstable global homotopy theory, most notably *topological stacks* and *orbispaces* as developed by Gepner and Henriques in their paper [61]. For a comparison of these two models to our orthogonal space model we refer to the author's paper [145]. The comparison proceeds through yet another model, the global homotopy theory of 'spaces with an action of the universal compact Lie group'. Here the universal compact Lie group (which is neither compact nor a Lie group) is the topological monoid $\mathcal{L}$ of linear isometric self-embeddings of $\mathbb{R}^\infty$, and in [145] we establish a global model structure on the category of $\mathcal{L}$-spaces.

1.1 Orthogonal spaces and global equivalences

In this section we introduce *orthogonal spaces* along with the notion of *global equivalences*, our setup to rigorously formulate the idea of 'compatible equivariant homotopy types for all compact Lie groups'. We introduce various basic techniques to manipulate global equivalences of orthogonal spaces, such as recognition criteria by homotopy or strict colimits over representations (Propositions 1.1.7 and 1.1.17), and a list of standard constructions that preserve global equivalences (Proposition 1.1.9). Theorem 1.1.10 is a cofinality result for orthogonal spaces showing that fairly general changes in the indexing category of linear isometries do not affect the global homotopy type. Definition 1.1.27 introduces global classifying spaces of compact Lie groups, the basic building blocks of global homotopy theory. Proposition 1.1.30 justifies the name by explaining the sense in which the global classifying space $B_{\mathrm{gl}}G$ 'globally classifies' principal G-bundles.

Before we start let us fix some notation and conventions. By a 'space' we mean a *compactly generated space* in the sense of [118], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff condition, see Definition A.1. We denote the category of compactly generated spaces by $\mathbf{T}$ and review its basic properties in Appendix A.

An *inner product space* is a finite-dimensional real vector space equipped with a scalar product, i.e., a positive-definite symmetric bilinear form. We denote by $\mathbf{L}$ the category with objects the inner product spaces and morphisms the linear isometric embeddings. The category $\mathbf{L}$ is a topological category in the sense that the morphism spaces come with a preferred topology: if $\varphi\colon V \longrightarrow W$ is a linear isometric embedding, then the action of the orthogonal group $O(W)$, by post-composition, induces a bijection

$$O(W)/O(\varphi^\perp) \cong \mathbf{L}(V,W) , \quad A \cdot O(\varphi^\perp) \longmapsto A \circ \varphi ,$$

where $\varphi^\perp = W - \varphi(V)$ is the orthogonal complement of the image of φ. We topologize $\mathbf{L}(V,W)$ so that this bijection is a homeomorphism, and this topology is independent of φ. If $(v_1, \ldots, v_k)$ is an orthonormal basis of V, then for every linear isometric embedding $\varphi\colon V \longrightarrow W$ the tuple $(\varphi(v_1), \ldots, \varphi(v_k))$ is an orthonormal k-frame of W. This assignment is a homeomorphism from $\mathbf{L}(V, W)$ to the Stiefel manifold of k-frames in W.

An example of an inner product space is the vector space $\mathbb{R}^n$ with the standard scalar product

$$\langle x, y\rangle \ = \ x_1y_1 \ + \ \cdots \ + \ x_ny_n \ .$$

In fact, every inner product space V is isometrically isomorphic to the inner product space $\mathbb{R}^n$, for n the dimension of V. So the full topological subcategory with the object $\mathbb{R}^n$ is a small skeleton of $\mathbf{L}$.

Definition 1.1.1 An *orthogonal space* is a continuous functor $Y\colon \mathbf{L} \longrightarrow \mathbf{T}$ to the category of spaces. A morphism of orthogonal spaces is a natural transformation. We denote the category of orthogonal spaces by *spc*.

The use of continuous functors from the category $\mathbf{L}$ to spaces has a long history in homotopy theory. The systematic use of inner product spaces (as opposed to numbers) to index objects in stable homotopy theory seems to go back to Boardman's thesis [15]. The category $\mathbf{L}$ (or its extension that also contains countably infinite-dimensional inner product spaces) is denoted $\mathscr{I}$ by Boardman and Vogt [16], and this notation is also used in [112]; other sources [102] use the symbol $\mathcal{I}$. Accordingly, orthogonal spaces are sometimes referred to as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces. Our justification for using yet another name is twofold: first, our use of orthogonal spaces comes with a shift in emphasis, away from a focus on non-equivariant homotopy types and towards viewing an orthogonal space as representing compatible equivariant homotopy types for all compact Lie groups. Second, we want to stress the analogy between orthogonal spaces and orthogonal spectra, the former being an unstable global world with the latter the corresponding stable global world.

Now we define our main new concept, the notion of 'global equivalence' between orthogonal spaces. We let G be a compact Lie group. By a *G-representation* we mean a finite-dimensional orthogonal representation, i.e., a real inner product space equipped with a continuous G-action by linear isometries. In other words, a G-representation consists of an inner product space V and a continuous homomorphism $\rho\colon G \longrightarrow O(V)$. In this context, and throughout the book, we will often use without explicit mention that continuous homomorphisms between Lie groups are automatically smooth, see for example [28, Prop. I.3.12]. For every orthogonal space Y and every G-representation V, the value $Y(V)$ inherits a G-action from the G-action on V and the functoriality of Y. For a G-equivariant linear isometric embedding $\varphi\colon V \longrightarrow W$, the induced map $Y(\varphi)\colon Y(V) \longrightarrow Y(W)$ is G-equivariant.

We denote by

$$D^k = \{x \in \mathbb{R}^k : \langle x, x\rangle \le 1\} \quad \text{and} \quad \partial D^k = \{x \in \mathbb{R}^k : \langle x, x\rangle = 1\}$$

the unit disc in $\mathbb{R}^k$ and its boundary, a sphere of dimension $k-1$, respectively. In particular, $D^0 = \{0\}$ is a one-point space and $\partial D^0 = \emptyset$ is empty.

Definition 1.1.2 A morphism $f: X \longrightarrow Y$ of orthogonal spaces is a *global equivalence* if the following condition holds: for every compact Lie group G, every G-representation V, every $k \ge 0$ and all continuous maps $\alpha: \partial D^k \longrightarrow X(V)^G$ and $\beta: D^k \longrightarrow Y(V)^G$ such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi: V \longrightarrow W$ and a continuous map $\lambda: D^k \longrightarrow X(W)^G$ such that $\lambda|_{\partial D^k} = X(\varphi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\varphi)^G \circ \beta$.

In other words, for every commutative square on the left

$$\begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[d, "f(V)^G"] \\ D^k \arrow[r, "\beta"'] & Y(V)^G \end{tikzcd} \qquad \begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[r, "X(\varphi)^G"] & X(W)^G \arrow[d, "f(W)^G"] \\ D^k \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G, \end{tikzcd}$$

there exists the lift λ on the right-hand side that makes the upper left triangle commute on the nose, and the lower right triangle commute up to homotopy relative to ∂D^k. In such a situation we will often refer to the pair (α, β) as a 'lifting problem' and we will say that the pair (φ, λ) *solves the lifting problem*.

Example 1.1.3 If $X = \underline{A}$ and $Y = \underline{B}$ are the constant orthogonal spaces with values A and B, and $f = \underline{g}$ is the constant morphism associated with a continuous map $g: A \longrightarrow B$, then $\underline{g}$ is a global equivalence if and only if for every commutative square

$$\begin{tikzcd} \partial D^k \arrow[r] \arrow[d, "\mathrm{incl}"'] & A \arrow[d, "g"] \\ D^k \arrow[r] \arrow[ur, dashed, "\lambda"] & B, \end{tikzcd}$$

there exists a lift λ that makes the upper left triangle commute, and the lower right triangle commute up to homotopy relative to ∂D^k. But this is one of the equivalent ways of characterizing weak equivalences of spaces, compare [114, Sec. 9.6, Lemma]. So $\underline{g}$ is a global equivalence if and only if g is a weak equivalence.

Remark 1.1.4 The notion of global equivalence is meant to capture the idea that for every compact Lie group G, some induced morphism

$$\mathrm{hocolim}_V\, f(V): \ \mathrm{hocolim}_V\, X(V) \longrightarrow \mathrm{hocolim}_V\, Y(V)$$

is a G-weak equivalence, where ‘$\mathrm{hocolim}_V$’ is a suitable homotopy colimit over all G-representations V along all equivariant linear isometric embeddings. This is a useful way to think about global equivalences and it could be made precise by letting V run over the poset of finite-dimensional subrepresentations of a complete G-universe and using the Bousfield–Kan construction of a homotopy colimit over this poset. Since the ‘poset of all G-representations’ has a cofinal subsequence, called an *exhaustive sequence* in Definition 1.1.6, we can also model the ‘homotopy colimit over all G-representations’ as the mapping telescope over an exhaustive sequence. However, the actual definition we work with has the advantage that it does not refer to universes and we do not have to define or manipulate homotopy colimits.

In many examples of interest all the structure maps of an orthogonal space Y are closed embeddings. When this is the case, the actual colimit (over the subrepresentations of a complete universe) of the G-spaces $Y(V)$ serves the purpose of a ‘homotopy colimit over all representations’ and it can be used to detect global equivalences, compare Proposition 1.1.17 below.

We will now establish some useful criteria for detecting global equivalences. We call a continuous map $f\colon A \longrightarrow B$ an *h-cofibration* if it has the homotopy extension property, i.e., given a continuous map $\varphi\colon B \longrightarrow X$ and a homotopy $H\colon A \times [0,1] \longrightarrow X$ starting with φf, there is a homotopy $\bar{H}\colon B \times [0,1] \longrightarrow X$ starting with φ such that $\bar{H}\circ(f\times[0,1]) = H$. Below we will write $H_t = H(-,t)\colon A \longrightarrow X$. All h-cofibrations in the category of compactly generated spaces are closed embeddings, compare Proposition A.31. The following somewhat technical lemma should be well known, but I was unable to find a reference.

Lemma 1.1.5 *Let A be a subspace of a space B such that the inclusion $A \longrightarrow B$ is an h-cofibration. Let $f\colon X \longrightarrow Y$ be a continuous map and*

$$H\colon A \times [0,1] \longrightarrow X \qquad \text{and} \qquad K\colon B \times [0,1] \longrightarrow Y$$

homotopies such that $K|_{A\times[0,1]} = fH$. Then the lifting problem (H_0, K_0) has a solution if and only if the lifting problem (H_1, K_1) has a solution.

Proof The problem is symmetric, so we only show one direction. We suppose that the lifting problem (H_0, K_0) has a solution consisting of a continuous map $\lambda\colon B \longrightarrow X$ such that $\lambda|_A = H_0$ and a homotopy $G\colon B \times [0,1] \longrightarrow Y$ such that

$$G_0 = f\circ\lambda\,, \quad G_1 = K_0 \qquad \text{and} \qquad (G_t)|_A = f\circ H_0$$

for all $t \in [0,1]$. The homotopy extension property provides a homotopy $H'\colon B \times [0,1] \longrightarrow X$ such that

$$H'_0 = \lambda \qquad \text{and} \qquad H'|_{A\times[0,1]} = H\,.$$

Then the map $\lambda' = H'_1 \colon B \longrightarrow X$ satisfies

$$\lambda'|_A = (H'_1)|_A = H_1 .$$

We define a continuous map $J \colon B \times [0,3] \longrightarrow Y$ by

$$J_t = \begin{cases} f \circ H'_{1-t} & \text{for } 0 \le t \le 1, \\ G_{t-1} & \text{for } 1 \le t \le 2, \text{ and} \\ K_{t-2} & \text{for } 2 \le t \le 3. \end{cases}$$

In particular,

$$J_0 = f \circ \lambda' \qquad \text{and} \qquad J_3 = K_1;$$

so J almost witnesses the fact that λ' solves the lifting problem (H_1, K_1), except that J is *not* a relative homotopy.

We improve J to a relative homotopy from $f \circ \lambda'$ to K_1. We define a continuous map $L \colon A \times [0,3] \times [0,1] \longrightarrow Y$ by

$$L(-,t,s) = \begin{cases} f \circ H_{1-t} & \text{for } 0 \le t \le s, \\ f \circ H_{1-s} & \text{for } s \le t \le 3-s, \text{ and} \\ f \circ H_{t-2} & \text{for } 3-s \le t \le 3. \end{cases}$$

Then $L(-,-,0)$ is the constant homotopy at the map $f \circ H_1$, and

$$L(-,-,1) = J|_{A \times [0,3]} \colon A \times [0,3] \longrightarrow Y .$$

Since the inclusion of A into B is an h-cofibration, the inclusion of $B \times \{0\} \cup_{A \times \{0\}} A \times [0,1]$ into $B \times [0,1]$ has a continuous retraction; hence the inclusion

$$B \times \{0\} \times [0,1] \cup_{A \times \{0\} \times [0,1]} A \times [0,1] \times [0,1] \longrightarrow B \times [0,1] \times [0,1]$$

also has a continuous retraction. We abbreviate $D = [0,3] \times \{1\} \cup \{0,3\} \times [0,1]$; the pair of spaces $([0,3] \times [0,1], D)$ is pair-homeomorphic to $([0,1] \times [0,1], \{0\} \times [0,1])$. So the inclusion

$$B \times D \cup_{A \times D} A \times [0,3] \times [0,1] \longrightarrow B \times [0,3] \times [0,1]$$

has a continuous retraction. The map L and the map

$$J \cup \mathrm{const}_{f\lambda} \cup \mathrm{const}_{K_1} \colon B \times D = B \times ([0,3] \times \{1\} \cup \{0,3\} \times [0,1]) \longrightarrow Y$$

agree on $A \times D$, so there is a continuous map $\bar{L} \colon B \times [0,3] \times [0,1] \longrightarrow Y$ such that

$$\bar{L}(-,-,1) = J , \qquad \bar{L}|_{A \times [0,3] \times [0,1]} = L ,$$

and

$$\bar{L}(-,0,s) = f \circ \lambda \qquad \text{and} \qquad \bar{L}(-,1,s) = K_1$$

for all $s \in [0,1]$. The map $\bar{J} = \bar{L}(-,-,0)\colon B \times [0,3] \longrightarrow Y$ then satisfies

$$\bar{J}|_{A\times[0,3]} \;=\; \bar{L}(-,-,0)|_{A\times[0,3]} \;=\; L(-,-,0)\,,$$

which is the constant homotopy at the map $f \circ H_1$; so $\bar{J}$ is a homotopy (parametrized by $[0,3]$ instead of $[0,1]$) *relative* to A. Because

$$\bar{J}_0 \;=\; \bar{L}(-,0,0) \;=\; f \circ \lambda \qquad \text{and} \qquad \bar{J}_3 \;=\; \bar{L}(-,3,0) \;=\; K_1\,,$$

the homotopy $\bar{J}$ witnesses that λ' solves the lifting problem (H_1, K_1). □

Definition 1.1.6 Let G be a compact Lie group. An *exhaustive sequence* is a nested sequence

$$V_1 \;\subset\; V_2 \;\subset\; \cdots \;\subset\; V_n \;\subset\; \cdots$$

of finite-dimensional G-representations such that every finite-dimensional G-representation admits a linear isometric G-embedding into some V_n.

Given an exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations and an orthogonal space Y, the values at the representations and their inclusions form a sequence of G-spaces and G-equivariant continuous maps

$$Y(V_1) \;\longrightarrow\; Y(V_2) \;\longrightarrow\; \cdots \;\longrightarrow\; Y(V_i) \;\longrightarrow\; \cdots\,.$$

We denote by

$$\operatorname{tel}_i\, Y(V_i)$$

the mapping telescope of this sequence of G-spaces; this telescope inherits a natural G-action.

We recall that a G-equivariant continuous map $f\colon A \longrightarrow B$ between G-spaces is a G-*weak equivalence* if for every closed subgroup H of G, the map $f^H\colon A^H \longrightarrow B^H$ of H-fixed points is a weak homotopy equivalence (in the non-equivariant sense).

Proposition 1.1.7 *The following three conditions are equivalent for every morphism of orthogonal spaces $f\colon X \longrightarrow Y$.*

(i) *The morphism f is a global equivalence.*

(ii) *For every compact Lie group G, every G-representation V, every finite G-CW-pair (B,A) and all continuous G-maps $\alpha\colon A \longrightarrow X(V)$ and $\beta\colon B \longrightarrow Y(V)$ such that $\beta|_A = f(V)\circ\alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi\colon V \longrightarrow W$ and a continuous G-map $\lambda\colon B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi)\circ\alpha$ and such that $f(W)\circ\lambda$ is G-homotopic, relative to A, to $Y(\varphi)\circ\beta$.*

(iii) *For every compact Lie group G and every exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations, the induced map*

$$\mathrm{tel}_i\, f(V_i)\colon\ \mathrm{tel}_i\, X(V_i) \longrightarrow \mathrm{tel}_i\, Y(V_i)$$

is a G-weak equivalence.

Proof At various places in the proof we use without explicitly mentioning it that taking G-fixed points commutes with formation of the mapping telescopes; this follows from the fact that taking G-fixed points commutes with pushouts along closed embeddings and with sequential colimits along closed embeddings, compare Proposition B.1.

(i)$\Longrightarrow$(ii) We argue by induction over the number of relative G-cells in (B, A). If $B = A$, then $\lambda = \alpha$ solves the lifting problem and there is nothing to show. Now we suppose that A is a proper subcomplex of B. We choose a G-CW-subcomplex B' that contains A and such that (B, B') has exactly one equivariant cell. Then (B', A) has strictly fewer cells, and the restricted equivariant lifting problem $(\alpha\colon A \longrightarrow X(V), \beta' = \beta|_{B'}\colon B' \longrightarrow Y(V))$ has a solution $(\varphi\colon V \longrightarrow U, \lambda'\colon B' \longrightarrow X(U))$ by the inductive hypothesis.

We choose a characteristic map for the last cell, i.e., a pushout square of G-spaces

$$\begin{tikzcd} G/H \times \partial D^k \arrow[r, "\chi"] \arrow[d, "\mathrm{incl}"'] & B' \arrow[d, "\mathrm{incl}"] \\ G/H \times D^k \arrow[r, "\chi"'] & B \end{tikzcd}$$

in which H is a closed subgroup of G. We arrive at the non-equivariant lifting problem on the left:

$$\begin{tikzcd} \partial D^k \arrow[r, "(\lambda')^H \circ \bar\chi"] \arrow[d, "\mathrm{incl}"'] & X(U)^H \arrow[d, "f(U)^H"] \\ D^k \arrow[r, "Y(\varphi)^H \circ \beta^H \circ \bar\chi"'] & Y(U)^H \end{tikzcd} \qquad \begin{tikzcd} \partial D^k \arrow[r, "(\lambda')^H \circ \bar\chi"] \arrow[d, "\mathrm{incl}"'] & X(U)^H \arrow[r, "X(\psi)^H"] & X(W)^H \arrow[d, "f(W)^H"] \\ D^k \arrow[r, "Y(\varphi)^H \circ \beta^H \circ \bar\chi"'] \arrow[urr, dashed, "\lambda"] & Y(U)^H \arrow[r, "Y(\psi)^H"'] & Y(W)^H . \end{tikzcd}$$

Here $\bar\chi = \chi(eH, -)\colon D^k \longrightarrow B^H$. Since f is a global equivalence, there is an H-equivariant linear isometric embedding $\psi\colon U \longrightarrow W$ and a continuous map $\lambda\colon D^k \longrightarrow X(W)^H$ such that $\lambda|_{\partial D^k} = X(\psi)^H \circ (\lambda')^H \circ \bar\chi$ and $f(W)^H \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\psi)^H \circ Y(\varphi)^H \circ \beta^H \circ \bar\chi$, as illustrated by the diagram on the right above. By enlarging W if necessary we can assume without loss of generality that W is underlying a G-representation and even that ψ is G-equivariant.

The G-equivariant extension of λ

$$G/H \times D^k \longrightarrow X(W)\,, \quad (gH, x) \longmapsto g \cdot \lambda(x)$$

and the map $X(\psi)\circ\lambda'\colon B' \longrightarrow X(W)$ then agree on $G/H\times\partial D^k$, so they glue to a G-map $\tilde\lambda\colon B \longrightarrow X(W)$. The pair $(\psi\varphi\colon V \longrightarrow W, \tilde\lambda\colon B \longrightarrow X(W))$ then solves the original lifting problem (α,β).

(ii)$\Longrightarrow$(iii) We suppose that f satisfies (ii) and we let G be any compact Lie group and $\{V_i\}_{i\geq 1}$ an exhaustive sequence of G-representations. We consider an equivariant lifting problem, i.e., a finite G-CW-pair (B,A) and a commutative square:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & \mathrm{tel}_i\, X(V_i) \arrow[d, "\mathrm{tel}_i\, f(V_i)"] \\ B \arrow[r, "\beta"'] & \mathrm{tel}_i\, Y(V_i)\ . \end{tikzcd}$$

We show that every such lifting problem has an equivariant solution. Since B and A are compact, there is an $n\geq 0$ such that α has image in the truncated telescope $\mathrm{tel}_{[0,n]}\, X(V_i)$ and β has image in the truncated telescope $\mathrm{tel}_{[0,n]}\, Y(V_i)$ (see Proposition A.15 (i)). There is a natural equivariant homotopy from the identity of the truncated telescope $\mathrm{tel}_{[0,n]}\, X(V_i)$ to the composite

$$\mathrm{tel}_{[0,n]}\, X(V_i) \xrightarrow{\mathrm{proj}} X(V_n) \xrightarrow{\mathrm{incl}} \mathrm{tel}_{[0,n]}\, X(V_i)\ .$$

Naturality means that this homotopy is compatible with the same homotopy for the telescope of the G-spaces $Y(V_i)$. Lemma 1.1.5 (or rather its G-equivariant generalization) applies to these homotopies, so instead of the original lifting problem we may solve the homotopic lifting problem

$$\begin{tikzcd} A \arrow[r, "\alpha'"] \arrow[d, "\mathrm{incl}"'] & X(V_n) \arrow[r, "i_n"] \arrow[d, "f(V_n)"] & \mathrm{tel}_i\, X(V_i) \arrow[d, "\mathrm{tel}_i\, f(V_i)"] \\ B \arrow[r, "\beta'"'] & Y(V_n) \arrow[r, "i_n"'] & \mathrm{tel}_i\, Y(V_i), \end{tikzcd}$$

where α' is the composite of the projection $\mathrm{tel}_{[0,n]}\, X(V_i) \longrightarrow X(V_n)$ with α, viewed as a map into the truncated telescope, and similarly for β'.

Since f satisfies (ii), the lifting problem $(\alpha'\colon A \longrightarrow X(V_n), \beta'\colon B \longrightarrow Y(V_n))$ has a solution after enlarging V_n along some linear isometric G-embedding. Since the sequence $\{V_i\}_{i\geq 1}$ is exhaustive, we can take this embedding as the inclusion $i\colon V_n \longrightarrow V_m$ for some $m\geq n$, i.e., there is a continuous G-map $\lambda\colon B \longrightarrow X(V_m)$ such that $\lambda|_A = X(i)^G\circ\alpha'$ and such that $f(V_m)^G\circ\lambda$ is

G-homotopic, relative to A, to $Y(i)^G \circ \beta'$, compare the diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha'"] \arrow[d, "\mathrm{incl}"'] & X(V_n) \arrow[r, "X(i)"] & X(V_m) \arrow[d, "f(V_m)"] \\ B \arrow[r, "\beta'"'] \arrow[urr, dashed, "\lambda"] & Y(V_n) \arrow[r, "Y(i)"'] & Y(V_m). \end{tikzcd}$$

The composite

$$X(V_n) \xrightarrow{X(i)} X(V_m) \xrightarrow{i_m} \mathrm{tel}_i\, X(V_i)$$

does *not* agree with $i_n \colon X(V_n) \longrightarrow \mathrm{tel}_i\, X(V_i)$, so the composite $i_n \circ \lambda \colon B \longrightarrow \mathrm{tel}_i\, X(V_i)$ does not quite solve the (modified) lifting problem $(i_n \circ \alpha', i_n \circ \beta')$. But there is a G-equivariant homotopy $H \colon X(V_n) \times [0,1] \longrightarrow \mathrm{tel}_i\, X(V_i)$ between $i_m \circ X(i)$ and i_n and a similar homotopy $K \colon Y(V_n) \times [0,1] \longrightarrow \mathrm{tel}_i\, Y(V_i)$ for Y instead of X. These homotopies satisfy

$$K \circ (f(V_n) \times [0,1]) \;=\; (\mathrm{tel}_i\, f(V_i)) \circ H\;,$$

so Lemma 1.1.5 implies that the modified lifting problem, and hence the original lifting problem, has an equivariant solution.

(iii)$\Longrightarrow$(i) We let G be a compact Lie group, V a G-representation, $k \geq 0$ and $(\alpha \colon \partial D^k \longrightarrow X(V)^G,\ \beta \colon D^k \longrightarrow Y(V)^G)$ a lifting problem, i.e., such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$. We choose an exhaustive sequence $\{V_i\}$ of G-representations; then we can embed V into some V_n by a linear isometric G-map and thereby assume without loss of generality that $V = V_n$.

We let $i_n \colon X(V_n) \longrightarrow \mathrm{tel}_i\, X(V_i)$ and $i_n \colon Y(V_n) \longrightarrow \mathrm{tel}_i\, Y(V_i)$ be the canonical maps. Since $\mathrm{tel}_i\, f(V_i) : \mathrm{tel}_i\, X(V_i) \longrightarrow \mathrm{tel}_i\, Y(V_i)$ is a G-weak equivalence, there is a continuous map $\lambda \colon D^k \longrightarrow (\mathrm{tel}_i\, X(V_i))^G$ such that $\lambda|_{\partial D^k} = i_n^G \circ \alpha$ and $(\mathrm{tel}_i\, f(V_i))^G \circ \lambda$ is homotopic, relative to ∂D^k, to $i_n^G \circ \beta$. Since fixed points commute with mapping telescopes, and since D^k is compact, there is an $m \geq n$ such that λ and the relative homotopy that witnesses the relation $(\mathrm{tel}_i\, f(V_i))^G \circ \lambda \simeq i_n^G \circ \beta$ both have image in $\mathrm{tel}_{[0,m]}\, X(V_i)^G$, the truncated telescope of the G-fixed points. The following diagram commutes

$$\begin{tikzcd} X(V_n)^G \arrow[r, "\mathrm{can}"'] \arrow[d, "f(V_n)^G"'] \arrow[rrr, bend left, "X(\mathrm{incl})^G"] & \mathrm{tel}_{[0,n]}\, X(V_i)^G \arrow[r, "\mathrm{incl}"'] \arrow[d, "\mathrm{tel}\, f(V_i)^G"'] & \mathrm{tel}_{[0,m]}\, X(V_i)^G \arrow[r, "\mathrm{proj}"'] \arrow[d, "\mathrm{tel}\, f(V_i)^G"] & X(V_m)^G \arrow[d, "f(V_m)^G"] \\ Y(V_n)^G \arrow[r, "\mathrm{can}"] \arrow[rrr, bend right, "Y(\mathrm{incl})^G"'] & \mathrm{tel}_{[0,n]}\, Y(V_i)^G \arrow[r, "\mathrm{incl}"] & \mathrm{tel}_{[0,m]}\, Y(V_i)^G \arrow[r, "\mathrm{proj}"] & Y(V_m)^G, \end{tikzcd}$$

where the right horizontal maps are the projections of the truncated telescope to