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978-1-108-42579-7 — A Short Course in Differential Topology
Bjørn Ian Dundas
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A Short Course in Differential Topology

Manifolds abound in mathematics and physics, and increasingly in cybernetics and visualization, where they often reflect properties of complex systems and their configurations. Differential topology gives us tools to study these spaces and extract information about the underlying systems.

This book offers a concise and modern introduction to the core topics of differential topology for advanced undergraduates and beginning graduate students. It covers the basics on smooth manifolds and their tangent spaces before moving on to regular values and transversality, smooth flows and differential equations on manifolds, and the theory of vector bundles and locally trivial fibrations. The final chapter gives examples of local-to-global properties, a short introduction to Morse theory and a proof of Ehresmann's fibration theorem.

The treatment is hands-on, including many concrete examples and exercises woven into the text, with hints provided to guide the student.

Bjørn Ian Dundas is Professor in the Mathematics Department at the University of Bergen. Besides his research and teaching he is the author of three books. When not doing mathematics he enjoys hiking and fishing from his rowboat in northern Norway.

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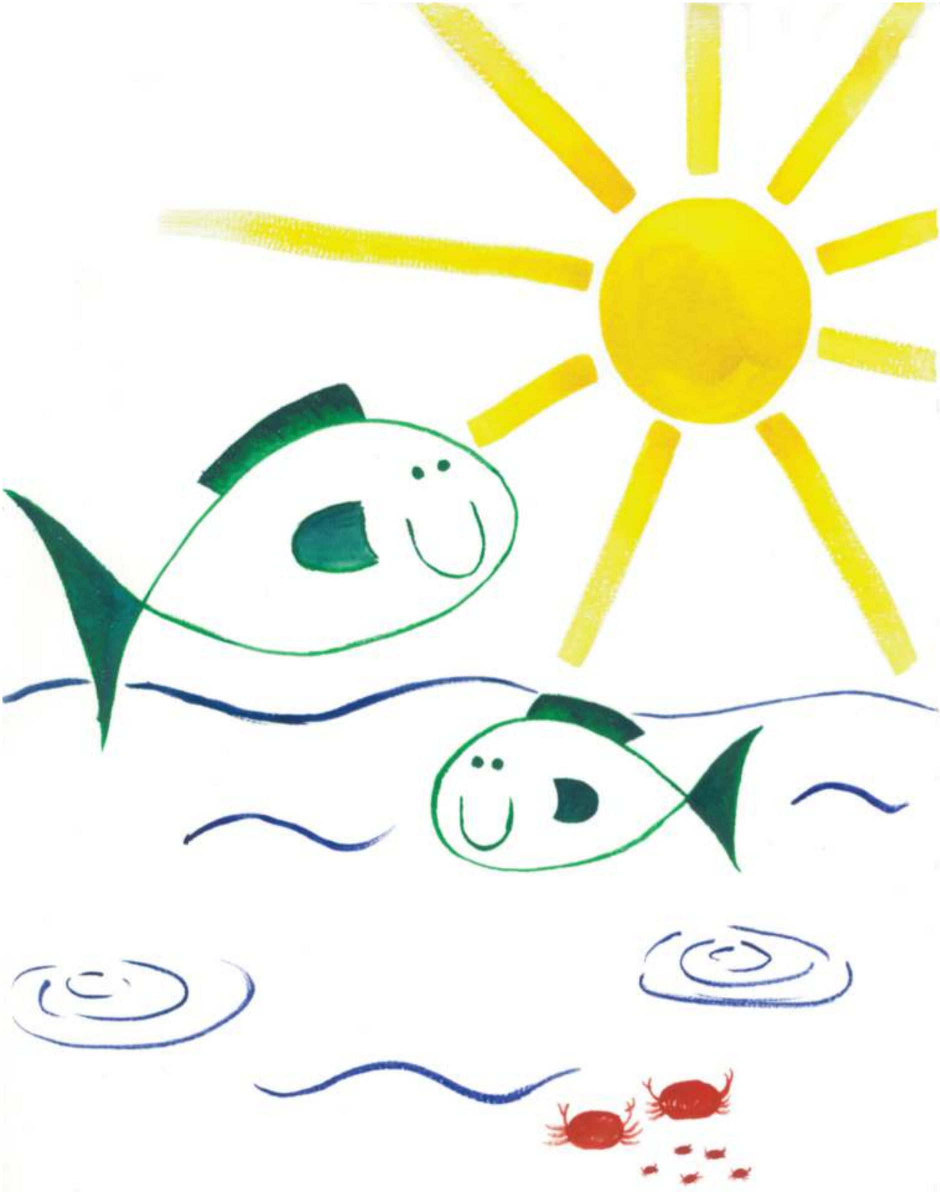

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Preface

In his inaugural lecture in 1854[1], Riemann introduced the concept of an "n-fach ausgedehnte Grösse" – roughly something that has "n degrees of freedom" and which we now would call an n-dimensional manifold.

Examples of manifolds are all around us and arise in many applications, but formulating the ideas in a satisfying way proved to be a challenge inspiring the creation of beautiful mathematics. As a matter of fact, much of the mathematical language of the twentieth century was created with manifolds in mind.

Modern texts often leave readers with the feeling that they are getting the answer before they know there is a problem. Taking the historical approach to this didactic problem has several disadvantages. The pioneers were brilliant mathematicians, but still they struggled for decades getting the concepts right. We must accept that we are standing on the shoulders of giants.

The only remedy I see is to give carefully chosen examples to guide the mind to ponder over the questions that you would actually end up wondering about even after spending a disproportionate amount of time. In this way I hope to encourage readers to appreciate and internalize the solutions when they are offered.

These examples should be concrete. On the other end of the scale, proofs should also be considered as examples: they are examples of successful reasoning. "Here is a way of handling such situations!" However, no amount of reading can replace doing, so there should be many opportunities for trying your hand.

In this book I have done something almost unheard of: I provide (sometimes quite lengthy) hints for all the exercises. This requires quite a lot of self-discipline from the reader: it is *very* hard not to peek at the solution too early. There are several reasons for including hints. First and foremost, the exercises are meant to be an integral part of class life. The exercises can be assigned to students who present their solutions in problem sessions, in which case the students must internalize their solution, but at the same time should be offered some moral support to lessen the social stress. Secondly, the book was designed for students who – even if eager to learn – are in need of more support with respect to how one can reason about the material. Trying your hand on the problem, getting stuck, taking a peek to see whether you glimpse an idea, trying again … and eventually getting a solution that you believe in and which you can discuss in class is *way* preferable to not having

[1] https://en.wikipedia.org/wiki/Bernhard_Riemann

anything to bring to class. A side effect is that this way makes it permissible to let the students develop parts of the text themselves without losing accountability. Lastly, though this was not a motivation for me, providing hints makes the text better suited for self-study.

Why This Book?

The year I followed the manifold course (as a student), we used Spivak [20], and I came to love the "Great American Differential Geometry book". At the same time, I discovered a little gem by Bröker and Jänich [4] in the library that saved me on some of the occasions when I got totally befuddled. I spent an inordinate amount of time on that class.

Truth be told, there are many excellent books on manifolds out there; to name just three, Lee's book [13] is beautiful; in a macho way so is Kosinski's [11]; and Milnor's pearl [15] will take you all the way from zero to framed cobordisms in 50 pages. Why write one more?

Cambridge University Press wanted "A Short Introduction" to precede my original title "Differential Topology". They were right: this is a far less ambitious text than the ones I have mentioned, and was designed for the students who took my classes. As a student I probably could provide a proof for all the theorems, but if someone asked me to check a very basic fact like "Is this map smooth?" I would feel that it was so for "obvious reasons" and hope for the life of me that no one would ask "why?" The book offers a modern framework while not reducing everything to some sort of magic. This allows us to take a hands-on approach; we are less inclined to identify objects without being specific about *how* they should be identified, removing some of the anxiety about "variables" and "coordinates changing" this or that way.

Spending time on the basics but still aiming at a one-semester course forces some compromises on this fairly short book. Sadly, topics like Sard's theorem, Stokes' theorem, differential forms, de Rham cohomology, differential equations, Riemannian geometry and surfaces, imbedding theory, K-theory, singularities, foliations and analysis on manifolds are barely touched upon.

At the end of the term, I hope that the reader will have internalized the fundamental ideas and will be able to use the basic language and tools with enough confidence to apply them in other fields, and to embark on more ambitious texts. Also, I wanted to prove Ehresmann's fibration theorem because I think it is cool.

How to Start Reading

The core curriculum consists of Chapters 2–8. The introduction in Chapter 1 is not strictly necessary for highly motivated readers who cannot wait to get to the theory, but provides some informal examples and discussions meant to put the later material into some perspective. If you are weak on point set topology, you will probably want to read Appendix A in parallel with Chapter 2. You should also be aware of the fact that Chapters 4 and 5 are largely independent, and, apart from a few exercises, can be read in any order. Also, at the cost of removing some exercises and examples, the sections on derivations (Section 3.5), orientations (Section 6.7), the generalized Gauss map (Section 6.8), second-order differential equations (Section 7.4), the exponential map (Section 8.2.7) and Morse theory (Section 8.4) can be removed from the curriculum without disrupting the logical development of ideas. The cotangent space/bundle material (Sections 3.4 and 5.6) can be omitted at the cost of using the dual tangent bundle from Chapter 6 onward.

Do the exercises, and only peek(!) at the hints if you *really* need to.

Prerequisites

Apart from relying on standard courses in multivariable analysis and linear algebra, this book is designed for readers who have already completed either a course in analysis that covers the basics of metric spaces or a first course in general topology. Most students will feel that their background in linear algebra could have been stronger, but it is to be hoped that seeing it used will increase their appreciation of things beyond Gaussian elimination.

Acknowledgments

First and foremost, I am indebted to the students and professors who have used the online notes and given me invaluable feedback. Special thanks go to Håvard Berland, Elise Klaveness, Torleif Veen, Karen Sofie Ronæss, Ivan Viola, Samuel Littig, Eirik Berge, Morten Brun and Andreas Leopold Knutsen. I owe much to a couple of anonymous referees (some unsolicited, but all very helpful) for their diligent reading and many constructive comments. They were probably right to insist that the final version should not actively insult the reader even if it means adopting less colorful language. The people at Cambridge University Press have all been very helpful, and I want especially to thank Clare Dennison, Tom Harris and Steven Holt.

My debt to the books [8], [11], [12], [13], [15], [14], [20] and in particular [4] should be evident from the text.

I am grateful to UiB for allowing me to do several revisions in an inspiring environment (Kistrand, northern Norway), and to the Hausdorff Institute in Bonn and the University of Copenhagen for their hospitality. The frontispiece is an adaption of one of my T-shirts. Thanks to Vår Iren Hjorth Dundas.

Notation

We let $\mathbf{N} = \{0, 1, 2, \ldots\}$, $\mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ be the sets of natural numbers, integers, rational numbers, real numbers and complex numbers. If X and Y are two sets, $X \times Y$ is the set of ordered pairs (x, y) with x an element in X and y an element in Y. If n is a natural number, we let $\mathbf{R}^n$ and $\mathbf{C}^n$ be the vector spaces of ordered n-tuples of real and complex numbers. Occasionally we

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may identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. If $p = (p_1, \dots, p_n) \in \mathbf{R}^n$, we let $|p|$ be the norm $\sqrt{p_1^2 + \cdots + p_n^2}$. The sphere of dimension n is the subset $S^n \subseteq \mathbf{R}^{n+1}$ of all $p = (p_0, \dots, p_n) \in \mathbf{R}^{n+1}$ with $|p| = 1$ (so that $S^0 = \{-1, 1\} \subseteq \mathbf{R}$, and S^1 can be viewed as all the complex numbers $e^{i\theta}$ of unit length).

Given functions $f : X \to Y$ and $g : Y \to Z$, we write gf for the composite, and $g \circ f$ only if the notation is cluttered and the $\circ$ improves readability. The constellation $g \cdot f$ will occur in the situation where f and g are functions with the same source and target, and where multiplication makes sense in the target. If X and Y are topological spaces, a continuous function $f : X \to Y$ is simply referred to as a *map*.