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Excerpt
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1 Introduction

The earth is round. This may at one point have been hard to believe, but we have grown accustomed to it even though our everyday experience is that the earth is (fairly) flat. Still, the most effective way to illustrate it is by means of maps: a globe (Figure 1.1) is a very neat device, but its global(!) character makes it less than practical if you want to represent fine details.

This phenomenon is quite common: locally you can represent things by means of "charts", but the global character can't be represented by a single chart. You need an entire atlas, and you need to know how the charts are to be assembled, or, even better, the charts overlap so that we know how they all fit together. The mathematical framework for working with such situations is manifold theory. Before we start off with the details, let us take an informal look at some examples illustrating the basic structure.

1.1 A Robot's Arm

To illustrate a few points which will be important later on, we discuss a concrete situation in some detail. The features that appear are special cases of general phenomena, and the example should provide the reader with some déjà vu experiences later on, when things are somewhat more obscure.

Figure 1.1. A globe. Photo by DeAgostini/Getty Images.

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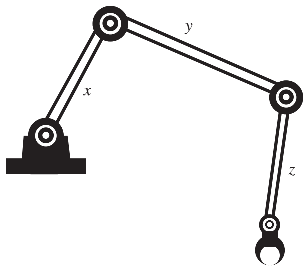

Figure 1.2.

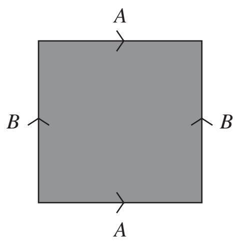

Figure 1.3.

Consider a robot's arm. For simplicity, assume that it moves in the plane, and has three joints, with a telescopic middle arm (see Figure 1.2).

Call the vector defining the inner arm x, that for the second arm y and that for the third arm z. Assume $|x| = |z| = 1$ and $|y| \in [1, 5]$. Then the robot can reach anywhere inside a circle of radius 7. But most of these positions can be reached in several different ways.

In order to control the robot optimally, we need to understand the various configurations, and how they relate to each other.

As an example, place the robot at the origin and consider all the possible positions of the arm that reach the point $P = (3, 0) \in \mathbf{R}^2$, i.e., look at the set T of all triples $(x, y, z) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$ such that

$$x + y + z = (3, 0), \qquad |x| = |z| = 1, \qquad |y| \in [1, 5].$$

We see that, under the restriction $|x| = |z| = 1$, x and z can be chosen arbitrarily, and determine y uniquely. So T is "the same as" the set

$$\{(x, z) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid |x| = |z| = 1\}.$$

Seemingly, our space T of configurations resides in four-dimensional space $\mathbf{R}^2 \times \mathbf{R}^2 \cong \mathbf{R}^4$, but that is an illusion – the space is two-dimensional and turns out to be a familiar shape. We can parametrize x and z by angles if we remember to identify the angles 0 and 2π. So T is what you get if you consider the square $[0, 2\pi] \times [0, 2\pi]$ and identify the edges as in Figure 1.3. See

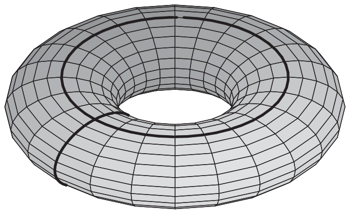

Figure 1.4.

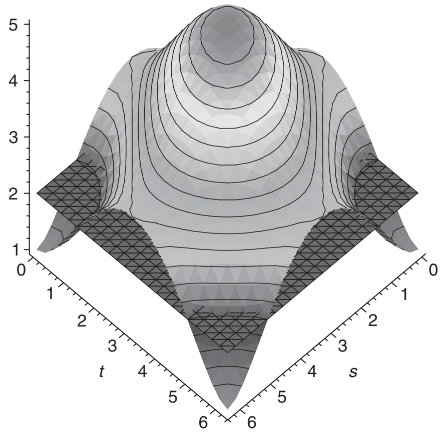

Figure 1.5.

`www.it.brighton.ac.uk/staff/jt40/MapleAnimations/Torus.html` for a nice animation of how the plane model gets glued.

In other words, the set T of all positions such that the robot reaches $P = (3, 0)$ may be identified with the torus in Figure 1.4. This is also true topologically in the sense that "close configurations" of the robot's arm correspond to points close to each other on the torus.

1.1.1 Question

What would the space S of positions look like if the telescope got stuck at $|y| = 2$?

Partial answer to the question: since $y = (3, 0) - x - z$ we could try to get an idea of what points of T satisfy $|y| = 2$ by means of inspection of the graph of $|y|$. Figure 1.5 is an illustration showing $|y|$ as a function of T given as a graph over $[0, 2\pi] \times [0, 2\pi]$, and also the plane $|y| = 2$.

The desired set S should then be the intersection shown in Figure 1.6. It looks a bit weird before we remember that the edges of $[0, 2\pi] \times [0, 2\pi]$ should be identified. On the torus it looks perfectly fine; and we can see this if we change our perspective a bit. In order to view T we chose $[0, 2\pi] \times [0, 2\pi]$ with identifications along the boundary. We could just as well have chosen $[-\pi, \pi] \times [-\pi, \pi]$, and then the picture would have looked like Figure 1.7. It does not touch the boundary,

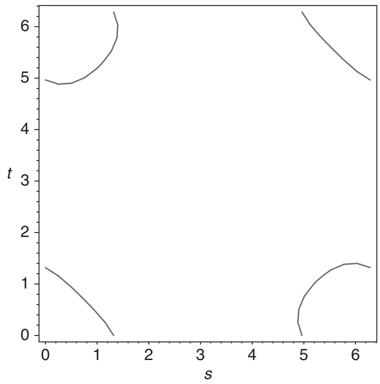

Figure 1.6.

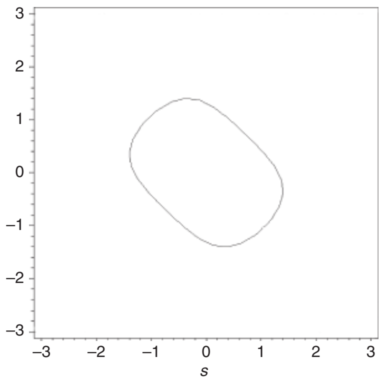

Figure 1.7.

so we do not need to worry about the identifications. As a matter of fact, S is homeomorphic to the circle (*homeomorphic* means that there is a bijection between S and the circle, and both the function from the circle to S and its inverse are continuous. See Definition A.2.8).

1.1.2 Dependence on the Telescope's Length

Even more is true: we notice that S looks like a smooth and nice curve. This will not happen for all values of $|y|$. The exceptions are $|y| = 1$, $|y| = 3$ and $|y| = 5$. The values 1 and 5 correspond to one-point solutions. When $|y| = 3$ we get a picture like Figure 1.8 (the solution really ought to touch the boundary).

We will learn to distinguish between such circumstances. They are qualitatively different in many aspects, one of which becomes apparent if we view the example shown in Figure 1.9 with $|y| = 3$ with one of the angles varying in $[0, 2\pi]$ while the other varies in $[-\pi, \pi]$. With this "cross" there is no way our solution space is homeomorphic to the circle. You can give an interpretation of the picture

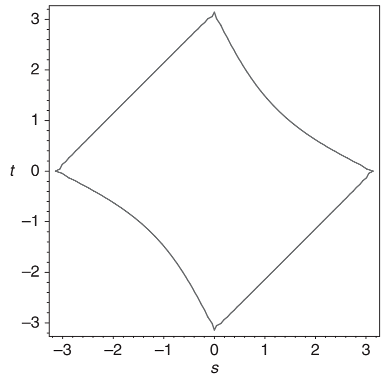

Figure 1.8.

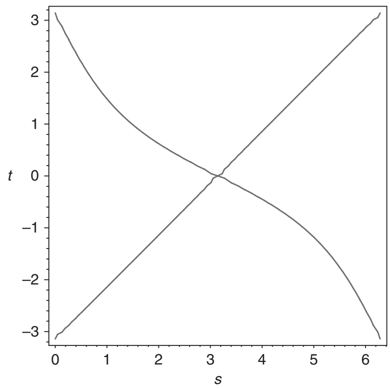

Figure 1.9.

above: the straight line is the movement you get if you let $x = z$ (like two wheels of equal radius connected by a coupling rod y on an old-fashioned train), whereas the curved line corresponds to x and z rotating in opposite directions (very unhealthy for wheels on a train).

Actually, this cross comes from a "saddle point" in the graph of $|y|$ as a function of T: it is a "critical" value at which all sorts of bad things can happen.

1.1.3 Moral

The configuration space T is smooth and nice, and we get different views on it by changing our "coordinates". By considering a function on T (in our case the length of y) and restricting to the subset of T corresponding to a given value of our function, we get qualitatively different situations according to what values we are looking at. However, away from the "critical values" we get smooth and nice subspaces, see in particular Theorem 4.4.3.

1.2 The Configuration Space of Two Electrons

Consider the situation where two electrons with the same spin are lonesome in space. To simplify matters, place the origin at the center of mass. The Pauli exclusion principle dictates that the two electrons cannot be at the same place, so the electrons are somewhere outside the origin diametrically opposite of each other (assume they are point particles). However, you can't distinguish the two electrons, so the only thing you can tell is what line they are on, and how far they are from the origin (you can't give a vector v saying that this points at a chosen electron: $-v$ is just as good).

Disregarding the information telling you how far the electrons are from each other (which anyhow is just a matter of scale), we get that the space of possible positions may be identified with the *space of all lines through the origin* in $\mathbf{R}^3$. This space is called the (real) projective plane $\mathbf{R}\mathrm{P}^2$. A line intersects the unit sphere $S^2 = \{p \in \mathbf{R}^3 \mid |p| = 1\}$ in exactly two (antipodal) points, and so we get that $\mathbf{R}\mathrm{P}^2$ can be viewed as the sphere S^2 **but** with $p \in S^2$ identified with $-p$. A point in $\mathbf{R}\mathrm{P}^2$ represented by $p \in S^2$ (and $-p$) is written $[p]$.

The projective plane is obviously a "manifold" (i.e., can be described by means of charts), since a neighborhood around $[p]$ can be identified with a neighborhood around $p \in S^2$ – as long as they are small enough to fit on one hemisphere. However, I cannot draw a picture of it in $\mathbf{R}^3$ without cheating.

On the other hand, there **is** a rather concrete representation of this space: it is what you get if you take a Möbius band (Figure 1.10) and a disk (Figure 1.11), and glue them together along their boundary (both the Möbius band and the disk have boundaries a copy of the circle). You are asked to perform this identification in Exercise 1.5.3.

1.2.1 Moral

The moral in this subsection is this: configuration spaces are oftentimes manifolds that do not in any natural way live in Euclidean space. From a technical point of view they often are what can be called *quotient spaces* (although this example was a rather innocent one in this respect).

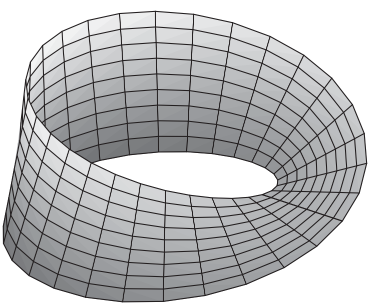

Figure 1.10. A Möbius band: note that its boundary is a circle.

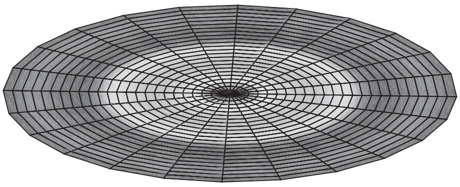

Figure 1.11. A disk: note that its boundary is a circle.

1.3 State Spaces and Fiber Bundles

The following example illustrates a phenomenon often encountered in physics, and a tool of vital importance for many applications. It is also an illustration of a key result which we will work our way towards: Ehresmann's fibration theorem, 8.5.10 (named after Charles Ehresmann, 1905–1979)[1].

It is slightly more involved than the previous example, since it points forward to many concepts and results we will discuss more deeply later, so if you find the going a bit rough, I advise you not to worry too much about the details right now, but come back to them when you are ready.

1.3.1 Qbits

In quantum computing one often talks about qbits. As opposed to an ordinary bit, which takes either the value 0 or the value 1 (representing "false" and "true" respectively), a *qbit*, or quantum bit, is represented by a complex linear combination ("superposition" in the physics parlance) of two states. The two possible states of a bit are then often called $|0\rangle$ and $|1\rangle$, and so a qbit is represented by the "pure qbit state" $\alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$ (since the total probability is 1, the numbers $|\alpha|^2$ and $|\beta|^2$ are interpreted as the probabilities that a measurement of the qbit will yield $|0\rangle$ and $|1\rangle$ respectively).

Note that the set of pairs $(\alpha, \beta) \in \mathbf{C}^2$ satisfying $|\alpha|^2 + |\beta|^2 = 1$ is just another description of the sphere $S^3 \subseteq \mathbf{R}^4 = \mathbf{C}^2$. In other words, a pure qbit state is a point (α, β) on the sphere S^3.

However, for various reasons *phase changes* are not important. A phase change is the result of multiplying $(\alpha, \beta) \in S^3$ by a unit-length complex number. That is, if $z = e^{i\theta} \in S^1 \subseteq \mathbf{C}$, the pure qbit state $(z\alpha, z\beta)$ is a phase shift of (α, β), and these should be identified. The *state space* is what you get when you identify each pure qbit state with the other pure qbit states you get by a phase change.

So, what is the relation between the space S^3 of pure qbit states and the state space? It turns out that the state space may be identified with the two-dimensional sphere S^2 (Figure 1.12), and the projection down to state space $\eta\colon S^3 \to S^2$ may then be given by

[1] https://en.wikipedia.org/wiki/Charles_Ehresmann

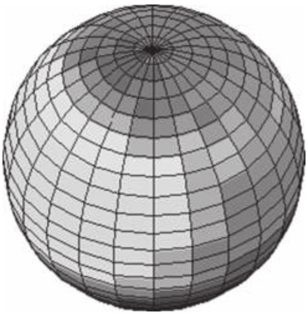

Figure 1.12. The state space S^2.

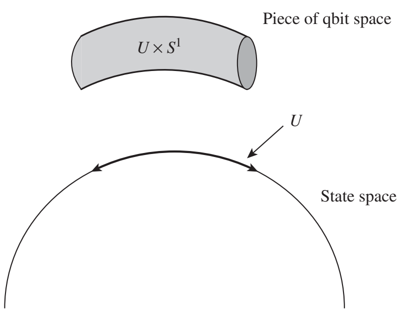

Figure 1.13. The pure qbit states represented in a small open neighborhood U in state space form a cylinder $U \times S^1$ (dimension reduced by one in the picture).

$$\eta(\alpha, \beta) = (|\alpha|^2 - |\beta|^2, 2\alpha\bar{\beta}) \in S^2 \subseteq \mathbf{R}^3 = \mathbf{R} \times \mathbf{C}.$$

Note that $\eta(\alpha, \beta) = \eta(z\alpha, z\beta)$ if $z \in S^1$, and so η does indeed send all the phase shifts of a given pure qbit to the same point in state space, and conversely, any two pure qbits in preimage of a given point in state space are phase shifts of each other.

Given a point in state space $p \in S^2$, the space of pure qbit states representing p can be identified with $S^1 \subseteq \mathbf{C}$: choose a pure qbit state (α, β) representing p, and note that any other pure qbit state representing p is of the form $(z\alpha, z\beta)$ for some *unique* $z \in S^1$.

So, can a pure qbit be given uniquely by its associated point in the state space and some point on the circle, i.e., is the space of pure qbit states really $S^2 \times S^1$ (and not S^3 as I previously claimed)? Without more work, it is not at all clear how these copies of S^1 lying over each point in S^2 are to be glued together: how does this "circle's worth" of pure qbit states change when we vary the position in state space slightly?

The answer comes through Ehresmann's fibration theorem, 8.5.10. It turns out that $\eta\colon S^3 \to S^2$ is a *locally* trivial fibration, which means that, in a small neighborhood U around any given point in state space, the space of pure qbit states *does* look like $U \times S^1$. See Figure 1.13. On the other hand, the *global* structure is different. In fact, $\eta\colon S^3 \to S^2$ is an important mathematical object for many reasons, and is known as the *Hopf fibration*.

The input to Ehresmann's theorem comes in two types. First we have some point set information, which in our case is handled by the fact that S^3 is "compact" A.7.1. Secondly, there is a condition which sees only the linear approximations, and which in our case boils down to the fact that any "infinitesimal" movement on S^2 is the shadow of an "infinitesimal" movement in S^3. This is a question which – given the right language – is settled through a quick and concrete calculation of differentials. We'll be more precise about this later (this is Exercise 8.5.16).

1.3.2 Moral

The idea is the important thing: if you want to understand some complicated model through some simplification, it is often so that the complicated model *locally* (in the simple model) can be built out of the simple model through multiplying with some fixed space.

How these local pictures are glued together to give the global picture is another matter, and often requires other tools, for instance from algebraic topology. In the $S^3 \to S^2$ case, we see that S^3 and $S^2 \times S^1$ cannot be identified since S^3 is simply connected (meaning that any closed loop in S^3 can be deformed continuously to a point) and $S^2 \times S^1$ is not.

An important class of examples (of which the above is one) of locally trivial fibrations arises from symmetries: if M is some (configuration) space and you have a "group of symmetries" G (e.g., rotations) acting on M, then you can consider the space M/G of points in M where you have identified two points in M if they can be obtained from each other by letting G act (e.g., one is a rotated copy of the other). Under favorable circumstances M/G will be a manifold and the projection $M \to M/G$ will be a locally trivial fibration, so that M is built by gluing together spaces of the form $U \times G$, where U varies over the open subsets of M/G.

1.4 Further Examples

A short bestiary of manifolds available to us at the moment might look like this.

- The surface of the earth, S^2, and higher-dimensional spheres, see Example 2.1.5.
- Space-time is a manifold: general relativity views space-time as a four-dimensional "pseudo-Riemannian" manifold. According to Einstein its curvature is determined by the mass distribution. (Whether the large-scale structure is flat or not is yet another question. Current measurements sadly seem to be consistent with a flat large-scale structure.)
- Configuration spaces in physics (e.g., the robot in Example 1.1, the two electrons of Example 1.2 or the more abstract considerations at the very end of Section 1.3.2 above).

- If $f: \mathbf{R}^n \to \mathbf{R}$ is a map and y a real number, then the inverse image

$$f^{-1}(y) = \{x \in \mathbf{R}^n \mid f(x) = y\}$$

  is often a manifold. For instance, if $f: \mathbf{R}^2 \to \mathbf{R}$ is the norm function $f(x) = |x|$, then $f^{-1}(1)$ is the unit circle S^1 (c.f. the discussion of submanifolds in Chapter 4).
- The torus (c.f. the robot in Example 1.1).
- "The real projective plane" $\mathbf{R}\mathrm{P}^2 = \{$All lines in $\mathbf{R}^3$ through the origin$\}$ (see the two-electron case in Example 1.2, but also Exercise 1.5.3).
- The Klein bottle[2] (see Section 1.5).

We end this introduction by studying surfaces in a bit more detail (since they are concrete, and this drives home the familiar notion of charts in more exotic situations), and also come up with some inadequate words about higher-dimensional manifolds in general.

1.4.1 Charts

The space-time manifold brings home the fact that manifolds must be represented intrinsically: the surface of the earth is seen as a sphere "in space", but there is no space which should naturally harbor the universe, except the universe itself. This opens up the question of how one can determine the shape of the space in which we live.

One way of representing the surface of the earth as the two-dimensional space it is (not referring to some ambient three-dimensional space), is through an atlas. The shape of the earth's surface is then determined by how each map in the atlas is to be glued to the other maps in order to represent the entire surface.

Just like the surface of the earth is covered by maps, the torus in the robot's arm was viewed through flat representations. In the technical sense of the word, the representation was not a "chart" (see Definition 2.1.1) since some points were covered twice (just as Siberia and Alaska have a tendency to show up twice on some European maps). It is allowed to have many charts covering Fairbanks in our atlas, but on each single chart it should show up at most once. We may fix this problem at the cost of having to use more overlapping charts. Also, in the robot example (as well as the two-electron and qbit examples) we saw that it was advantageous to operate with more charts.

Example 1.4.2 To drive home this point, please play Jeff Weeks' "Torus Games" on `www.geometrygames.org/TorusGames/` for a while.

[2] www-groups.dcs.st-and.ac.uk/~history/Biographies/Klein.html