1 Curves in $\mathbb{R}^n$

This book provides an account of the differential geometry of surfaces, principally (but not exclusively) in Euclidean 3-space. We shall be studying their metric geometry; both internal, or *intrinsic* geometry, and their external, or *extrinsic* geometry.

As a preliminary, in this chapter we study curves in the vector space $\mathbb{R}^n$ with its standard inner product. For the most part n will be 2 or 3 since we wish to emphasize the geometrical aspects in a way which can be easily visualized. The crucial properties of the curves we study are that they are 1-dimensional and may be approximated up to first order near any point by a straight line, the *tangent line* at that point. The intrinsic geometry of these curves is somewhat simple, consisting of the *arc length* along the curve between any two points on the curve, while the most important measure of the extrinsic geometry is the *curvature*, the rate at which the curve bends away from its tangent line.

The ideas in this chapter are important for what follows in the rest of the book for several reasons. Firstly, many of the ideas extend in a natural way to surfaces (and to the more general study of n-dimensional objects called *differentiable manifolds*), and so a number of important concepts are introduced here in the simplest possible situation. Secondly, the intrinsic and extrinsic geometry of a surface are most easily and intuitively studied by using curves on the surface. For instance, the geometry of a surface may be studied by means of its *geodesics*, which are the analogues for surfaces of straight lines in the plane. Finally, curves on a surface may often be regarded in a natural way as curves in the plane where this latter is now endowed with a non-standard metric, and many of the ideas we develop in this chapter may be extended to study this new situation.

There is a large and interesting body of work concerned with the local and global theory of curves in Euclidean space, but we have been rather ruthless in our selection of material. Other than the material on involutes and evolutes in §1.4 (some or all of which may be omitted if desired, since the material is not used directly in the rest of the book), we have restricted ourselves to those aspects of the theory that have most relevance to our study of surfaces.

The layout of the chapter is as follows. After some preliminary definitions and examples we consider the local theory of plane curves, where the notion of curvature is introduced. We then seek to give some familiarity with the ideas in the optional section on involutes and evolutes. Finally, we consider the local theory of space curves, where the behaviour is governed by two invariants, namely the curvature and the torsion.

1.1 Basic definitions

For each positive integer n, let $\mathbb{R}^n$ denote the n-dimensional vector space of n-tuples of real numbers, with vector addition and multiplication by a scalar λ carried out component-wise. Specifically,

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n) .$$

A *smooth parametrised curve* (henceforth called a *smooth curve*) in $\mathbb{R}^n$ is a smooth map $\boldsymbol{\alpha} : I \to \mathbb{R}^n$, where I is a possibly infinite open interval of real numbers. Thus $\boldsymbol{\alpha}(u) = (x_1(u), \ldots, x_n(u))$, where $x_1, \ldots, x_n : I \to \mathbb{R}$, are infinitely differentiable functions of u. The variable u is called the *parameter* and the image $\boldsymbol{\alpha}(I) \subset \mathbb{R}^n$ is called the *trace* of $\boldsymbol{\alpha}$. Intuitively, we are thinking of a curve as the path traced out by a point moving in $\mathbb{R}^n$.

The metric properties of such a curve (or indeed a surface) are derived from the metric properties of the containing Euclidean space $\mathbb{R}^n$. These are determined by the *inner product* (also called the *scalar* or *dot* product) on $\mathbb{R}^n$ which assigns to each pair of vectors $\boldsymbol{v} = (v_1, \ldots, v_n)$, $\boldsymbol{w} = (w_1, \ldots, w_n)$ the scalar $\boldsymbol{v}.\boldsymbol{w}$ given by

$$\boldsymbol{v}.\boldsymbol{w} = v_1 w_1 + \cdots + v_n w_n .$$

The *length* $|\boldsymbol{v}|$ of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is defined by $|\boldsymbol{v}| = \sqrt{\boldsymbol{v}.\boldsymbol{v}}$, and the *angle* θ between two non-zero vectors $\boldsymbol{v}$, $\boldsymbol{w}$ is given by

$$\boldsymbol{v}.\boldsymbol{w} = |\boldsymbol{v}|\,|\boldsymbol{w}| \cos\theta , \quad 0 \le \theta \le \pi .$$

We let $x'(u)$ denote the derivative of a function $x(u)$. Then the *tangent vector* to a smooth curve $\boldsymbol{\alpha}$ at u is given by $\boldsymbol{\alpha}'(u) = \big(x_1'(u), \ldots, x_n'(u)\big)$. As mentioned at the start of the chapter, the crucial property of the curves we wish to study is that they may be approximated up to first order near any point by a straight line, the *tangent line*. For this reason, we shall for the most part consider *regular curves*; these are smooth curves for which $\boldsymbol{\alpha}'(u)$ is never zero. The tangent line is then the line though $\boldsymbol{\alpha}(u)$ in direction $\boldsymbol{\alpha}'(u)$, and the *unit tangent vector* $\boldsymbol{t}$ to $\boldsymbol{\alpha}$ (Figure 1.1) is given by

$$\boldsymbol{t} = \frac{\boldsymbol{\alpha}'}{|\boldsymbol{\alpha}'|} .$$

In the above, and elsewhere when no confusion should arise, we omit specific reference to the parameter u.

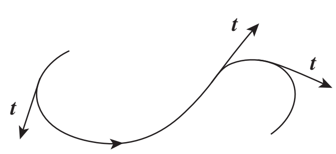

Figure 1.1 The trace of a regular curve

We shall often think of u as a time parameter, in which case $|\boldsymbol{\alpha}'|$ gives the *speed*, and $\boldsymbol{t}$ the *direction* of travel along $\boldsymbol{\alpha}$.

Example 1 (Ellipse) Let $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\boldsymbol{\alpha}(u) = (a \cos u, b \sin u)\,, \quad u \in \mathbb{R}\,,$$

where a and b are distinct positive real numbers. Then

$$\boldsymbol{\alpha}' = (-a \sin u, b \cos u) \neq \mathbf{0}\,,$$

so that

$$\boldsymbol{t} = \frac{(-a \sin u, b \cos u)}{(a^2 \sin^2 u + b^2 \cos^2 u)^{1/2}}\,,$$

and we see that $\boldsymbol{\alpha}$ is a regular curve whose trace is the ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

A point at which a smooth curve has vanishing derivative will be called a *singular point*.

Example 2 (Cusp point) Let $\boldsymbol{\alpha} : \mathbb{R} :\to \mathbb{R}^2$ be defined by

$$\boldsymbol{\alpha}(u) = (u^3, u^2)\,, \quad u \in \mathbb{R}\,.$$

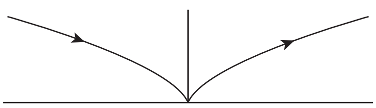

Figure 1.2 Cusp at $\boldsymbol{\alpha}(0)$

Then $\boldsymbol{\alpha}$ is smooth but not regular since $\boldsymbol{\alpha}'(0) = \mathbf{0}$. The trace of $\boldsymbol{\alpha}$ (Figure 1.2) is the curve $y^3 = x^2$ which has a cusp at $\boldsymbol{\alpha}(0)$. This is an example of the type of behaviour we exclude when we consider regular curves.

Of course, the restriction of the curve $\boldsymbol{\alpha}$ in Example 2 to $(0, \infty)$ and to $(-\infty, 0)$ are both regular curves, as is the restriction of any regular curve to an open subinterval of its domain of definition.

Example 3 (Helix) Let $\boldsymbol{\alpha} : \mathbb{R} :\to \mathbb{R}^3$ be defined by

$$\boldsymbol{\alpha}(u) = (a \cos u, a \sin u, bu)\,, \quad u \in \mathbb{R}\,,$$

where $a > 0$ and $b \neq 0$. Then

$$\boldsymbol{\alpha}' = (-a \sin u, a \cos u, b) \neq \mathbf{0}\,,$$

so that

$$\boldsymbol{t} = \frac{(-a \sin u, a \cos u, b)}{(a^2 + b^2)^{1/2}}\,,$$

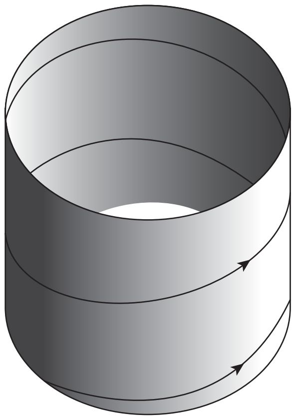

Figure 1.3 Helix on a cylinder

and we see that $\boldsymbol{\alpha}$ is a regular curve, called a *helix* (Figure 1.3); its trace lies on the cylinder $x^2 + y^2 = a^2$ in $\mathbb{R}^3$.

The *pitch* of the helix is $2\pi b$; this is the vertical distance between the points $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\alpha}(u + 2\pi)$, one point being obtained from the other after one complete revolution of the helix round the cylinder. We note that $|\boldsymbol{\alpha}'|$ is constant, so with this parametrisation we travel along the curve with constant speed.

Example 4 (Graph of a function) Let $g : I \to \mathbb{R}$ be a smooth function defined on an open interval I of real numbers. The *graph* $\Gamma(g)$ of g is the trace of the regular curve in $\mathbb{R}^2$ given by

$$\boldsymbol{\alpha}(u) = (u, g(u)) \ , \quad u \in I \ .$$

For example, the graph of $g(u) = u^2$ gives the parabola $y = x^2$.

The trace of the graph of a function g has equation $y - g(x) = 0$. It may be expected that a wealth of other examples may be written down using equations of the form $f(x, y) = c$, where c is constant and $f(x, y)$ is a smooth function of x and y. In fact, an equation of this type does not always give the trace of a regular curve (for instance $x^2 + y^2 = 0$, or, as we have seen, $y^3 = x^2$), and even when it does, we do not have a natural associated parameter. For these reasons, we discuss sets of points satisfying equations in the next chapter in the context of surfaces in $\mathbb{R}^3$.

We conclude this section with a slight extension of our treatment of curves. A smooth (*resp.* regular) curve on a **closed** interval $[a, b]$ is one which may be extended to a smooth (*resp.* regular) curve on an open interval containing $[a, b]$. A *closed* curve $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ is a regular curve such that $\boldsymbol{\alpha}$ and all its derivatives agree at the end points of the interval; that is,

$$\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b) \ , \quad \boldsymbol{\alpha}'(a) = \boldsymbol{\alpha}'(b) \ , \quad \boldsymbol{\alpha}''(a) = \boldsymbol{\alpha}''(b), \ \ldots \ .$$

For example, the restriction to $[-\pi, \pi]$ of the curve $\boldsymbol{\alpha}$ in Example 1 is a closed curve – it travels once round the ellipse, starting and ending at $(-a, 0)$.

1.2 Arc length

It is important to note that, as far as geometry is concerned, it is the **trace** (or **image**) of a smooth curve which is of interest; the parametrisation is just a convenient device for describing and studying this. A good choice of parametrisation is often helpful, however, as this can lead to a great simplification of a given problem. In this section we describe an intrinsic parametrisation for any regular curve; it is defined by taking the arc length in the direction of travel measured from some given point on the curve. This parametrisation is of fundamental importance in the general theory of regular curves but, as we shall indicate, finding such a parametrisation is impracticable for most examples and so is usually best avoided in explicit calculations.

Let $\boldsymbol{\alpha} : (a, b) \to \mathbb{R}^n$ be a smooth curve and let $u_0 \in (a, b)$. We define $s : (a, b) \to \mathbb{R}$ by integrating the speed of travel between $\boldsymbol{\alpha}(u_0)$ and $\boldsymbol{\alpha}(u)$. Thus

$$s(u) = \int_{u_0}^{u} |\boldsymbol{\alpha}'(v)| dv \tag{1.1}$$

is the *arc length* along $\boldsymbol{\alpha}$ measured from $\boldsymbol{\alpha}(u_0)$. Note that $s(u)$ is positive for $u > u_0$, and negative for $u < u_0$.

Example 1 (Ellipse) Let a and b be distinct positive real numbers and let $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ be the ellipse

$$\boldsymbol{\alpha}(u) = (a \cos u, b \sin u) , \quad u \in \mathbb{R} .$$

Then

$$\boldsymbol{\alpha}' = (-a \sin u, b \cos u) ,$$

and so, if $s(u)$ denotes arc length measured from $\boldsymbol{\alpha}(0)$, then

$$s(u) = \int_0^u \sqrt{a^2 \sin^2 v + b^2 \cos^2 v}\, dv .$$

This integral cannot be expressed in terms of elementary functions such as trigonometric functions, and serves to define a special class of functions called *elliptic functions*.

As the above example indicates, it may be difficult to write down explicit expressions in closed form (that is to say, in terms of standard functions) for functions describing the geometry, even in quite simple cases. In the following example, however, the calculations are all fairly straightforward.

Example 2 (Cycloid) This is the curve in the plane traced out by a point on a circle which rolls without slipping along a line (Figure 1.4).

Assuming that the radius of the circle is 1 and the circle rolls on the x-axis in $\mathbb{R}^2$, the curve may be parametrised by $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ where

$$\boldsymbol{\alpha}(u) = (u - \sin u, 1 - \cos u) .$$

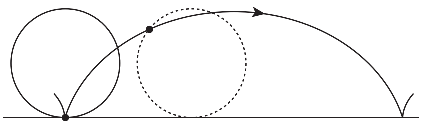

Figure 1.4 Cycloid

Then

$$\begin{aligned}\boldsymbol{\alpha}' &= (1-\cos u, \sin u)\\ &= \left(2\sin^2\frac{u}{2}, 2\sin\frac{u}{2}\cos\frac{u}{2}\right)\\ &= 2\sin\frac{u}{2}\left(\sin\frac{u}{2}, \cos\frac{u}{2}\right)\,,\end{aligned}$$

so that $\boldsymbol{\alpha}$ has singular points when $u = 2n\pi$, where n is an integer. These singular points correspond to the points where the cycloid touches the x-axis; at these points the cycloid has the characteristic cusp shape pointed out in Example 2 of §1.1.

Furthermore,

$$\begin{aligned}|\boldsymbol{\alpha}'| &= \left|2\sin\frac{u}{2}\right|\\ &= 2\sin\frac{u}{2}\,, \quad \text{for } 0 \le u \le 2\pi\,.\end{aligned}$$

Thus, for $0 \le u \le 2\pi$, if $s(u)$ denotes arc length measured from $\boldsymbol{\alpha}(0)$, then

$$\begin{aligned}s(u) &= \int_0^u 2\sin\frac{v}{2}dv\\ &= 4(1-\cos\frac{u}{2})\,.\end{aligned}$$

In particular, the length of a single arch of the cycloid is 8.

We now show that we may use arc length s to parametrise a regular curve, and describe some consequences of doing so. The most useful results we obtain are equation (1.4) and its immediate consequence that when we parametrise a regular curve by arc length we travel along it at unit speed.

We begin by noting that the arc length $s(u)$ along a regular curve $\boldsymbol{\alpha}(u)$ in $\mathbb{R}^n$ is a smooth function and, from (1.1),

$$\frac{ds}{du} = |\boldsymbol{\alpha}'| > 0\,. \tag{1.2}$$

Hence s is an increasing function of u, and we may use arc length to parametrise the trace of the curve in the same direction of travel. The chain rule for differentiation then tells us that

$$\frac{d}{du} = \frac{ds}{du}\frac{d}{ds}\,. \tag{1.3}$$

We now give a brief explanation of why (1.3) holds; this paragraph may be omitted by those who are happy with the chain rule as stated in (1.3). Let $\boldsymbol{\alpha}(u)$ be a regular curve, and parametrise it by arc length by letting $\tilde{\boldsymbol{\alpha}}(s)$ be the point on the trace of $\boldsymbol{\alpha}$ having arc length s from a chosen base point $\boldsymbol{\alpha}(u_0)$. Then $\boldsymbol{\alpha}(u) = \tilde{\boldsymbol{\alpha}}(s(u))$. More generally, given a function $\tilde{\boldsymbol{f}}(s)$, we let $\boldsymbol{f}(u) = \tilde{\boldsymbol{f}}(s(u))$. Then, since the derivative of a composite is the product of the derivatives,

$$\left.\frac{d\boldsymbol{f}}{du}\right|_u = \left.\frac{d\tilde{\boldsymbol{f}}}{ds}\right|_{s(u)} \left.\frac{ds}{du}\right|_u .$$

Following commonly used convention, we do not usually mention the points at which the differentiation takes place, and also, when there is no danger of confusion, we omit the $\tilde{}$ and simply write

$$\frac{d\boldsymbol{f}}{du} = \frac{d\boldsymbol{f}}{ds}\frac{ds}{du} ,$$

which gives the operator equation (1.3). This completes the optional paragraph of explanation of (1.3).

Returning to our account of the parametrisation of a regular curve using its arc length s, the chain rule (1.3), together with (1.2), shows that

$$\frac{d}{ds} = \frac{1}{|\boldsymbol{\alpha}'|}\frac{d}{du} , \tag{1.4}$$

and, in particular,

$$\frac{d\boldsymbol{\alpha}}{ds} = \frac{1}{|\boldsymbol{\alpha}'|}\boldsymbol{\alpha}' = \boldsymbol{t} , \tag{1.5}$$

so that when we parametrise a regular curve by arc length we travel along it at unit speed. With such a parametrisation, the arc length along $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}(s_0)$ to $\boldsymbol{\alpha}(s_1)$ is equal to $s_1 - s_0$.

Note that when, as above, we are considering two different parametrisations with the same trace, the notation $'$ for derivative must be used with care in order to avoid confusion between d/du and d/ds. We shall always use $'$ to denote d/du, the derivative with respect to the given parameter u of the curve, and we shall use d/ds to denote differentiation with respect to the arc length parameter.

We summarise the content of this section in the following theorem.

Theorem 3 *Let $\boldsymbol{\alpha}(u)$ be a regular curve in $\mathbb{R}^n$. Then we may parametrise the trace of $\boldsymbol{\alpha}$ using arc length s from a point $\boldsymbol{\alpha}(u_0)$ on $\boldsymbol{\alpha}$. If we do this, then $d\boldsymbol{\alpha}/ds$ is the unit tangent vector $\boldsymbol{t}$ to $\boldsymbol{\alpha}$ in the direction of travel. In particular, $\boldsymbol{t}$ is smoothly defined along $\boldsymbol{\alpha}$, and, when using arc length as parameter, we travel along $\boldsymbol{\alpha}$ at unit speed. The arc length along $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}(s_0)$ to $\boldsymbol{\alpha}(s_1)$ is equal to $s_1 - s_0$.*

It is important to note that if a curve is not regular then it cannot usually be parametrised by arc length past a singular point. For instance, the unit tangent vector in the direction of travel of the cycloid has discontinuities (and so is not smooth) at the singular points. A similar comment holds for the cusp curve in Example 2 of §1.1.

As mentioned at the start of this section, and as we shall see later, the existence of the arc length parameter is very important for theoretical work. However, arc length is not usually a good choice of parameter to use in calculations since in general it is difficult to find explicitly, as illustrated by Example 1.

1.3 The local theory of plane curves

In this section we introduce the signed curvature κ of a regular curve in the plane $\mathbb{R}^2$, which describes the way in which the curve is bending in the plane. We then discuss the fundamental theorem of the local theory of plane curves, which shows that a regular plane curve is determined essentially uniquely by its curvature as a function of arc length. In Chapter 6, we shall discuss Bonnet's Theorem, which is the analogous result for surfaces in $\mathbb{R}^3$.

The main goals of the first half of this section are to explain the moving frame equations (1.6) and (1.7), and to give examples of their use.

Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a regular curve defined on an open interval I, and, as usual, let d/ds denote differentiation with respect to arc length along $\boldsymbol{\alpha}$. As we have seen, the unit tangent vector is given by $\boldsymbol{t} = d\boldsymbol{\alpha}/ds$, and we let $\boldsymbol{n}$ be the unit vector obtained by rotating $\boldsymbol{t}$ anticlockwise through $\pi/2$. Thus, if $\boldsymbol{t} = (a, b)$ then $\boldsymbol{n} = (-b, a)$. Then $\{\boldsymbol{t}, \boldsymbol{n}\}$ is an *adapted orthonormal moving frame* along $\boldsymbol{\alpha}$ (Figure 1.5).

Since $\boldsymbol{t}.\boldsymbol{t} = 1$, we may use the product rule for differentiation to deduce that $\dfrac{d\boldsymbol{t}}{ds}.\boldsymbol{t} = 0$. Hence

$$\frac{d\boldsymbol{t}}{ds} = \kappa \boldsymbol{n} \tag{1.6}$$

for a uniquely determined smooth function κ called the *signed curvature* (or simply the *curvature*) of $\boldsymbol{\alpha}$. Similarly, $\dfrac{d\boldsymbol{n}}{ds}.\boldsymbol{n} = 0$, so that $\dfrac{d\boldsymbol{n}}{ds}$ is a scalar multiple of $\boldsymbol{t}$. Differentiating the expression $\boldsymbol{t}.\boldsymbol{n} = 0$ and applying (1.6) we see that

$$\frac{d\boldsymbol{n}}{ds} = -\kappa \boldsymbol{t} \,. \tag{1.7}$$

As we shall see, κ measures the rate of rotation of $\boldsymbol{t}$ (and $\boldsymbol{n}$) in an anticlockwise direction as we travel along the curve at unit speed.

Curvature is a measure of acceleration, and hence plays a big part in all our lives. For instance, it shows itself as the sideways force we, and our coffee cups(!), feel as we go round a bend on a railway train. When travelling at a given speed, the more the track bends, the quicker the coffee cup slides (or falls over, if the curvature is really big). When we are facing the direction of travel, the cup slides to our right if the curvature is positive, and to our left if it is negative.

Equations (1.6) and (1.7), which give the rate of change of each element of the moving frame $\{\boldsymbol{t}, \boldsymbol{n}\}$ in terms of the frame itself, are called the *moving frame equations*.

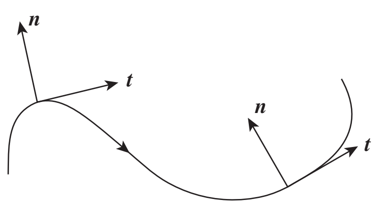

Figure 1.5 A moving frame

Example 1 (Circle of radius r) The circle with centre $\mathbf{a}$ and radius $r > 0$ traversed in an anticlockwise direction has constant curvature $\kappa = 1/r$. For, parametrising the circle by arc length, we have

$$\boldsymbol{\alpha}(s) = \mathbf{a} + r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right),$$

so that

$$\boldsymbol{t} = \frac{d\boldsymbol{\alpha}}{ds} = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right),$$

and

$$\boldsymbol{n} = -\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right).$$

Then

$$\frac{d\boldsymbol{t}}{ds} = -\frac{1}{r}\left(\cos\frac{s}{r}, \sin\frac{s}{r}u\right) = \frac{1}{r}\boldsymbol{n},$$

so that $\boldsymbol{\alpha}$ has curvature $1/r$. If the circle is traversed in a clockwise direction then it has curvature $-1/r$.

We now give an example to show how we may find the curvature of a regular curve $\boldsymbol{\alpha}$ which is not parametrised by arc length. In this, and much of the following, we repeatedly use equation (1.4). This equation will also be very useful in the following sections.

Example 2 (Cycloid) Recall from Example 2 in §1.2 that the cycloid may be parametrised as

$$\boldsymbol{\alpha}(u) = (u - \sin u, 1 - \cos u),$$

and that, using $'$ for d/du as usual,

$$\boldsymbol{\alpha}' = 2\sin\frac{u}{2}\left(\sin\frac{u}{2}, \cos\frac{u}{2}\right).$$

Hence, for $0 < u < 2\pi$,

$$\boldsymbol{t} = \left(\sin\frac{u}{2}, \cos\frac{u}{2}\right),$$
$$\boldsymbol{n} = \left(-\cos\frac{u}{2}, \sin\frac{u}{2}\right),$$
$$|\boldsymbol{\alpha}'| = 2\sin\frac{u}{2}.$$

Thus, using (1.4),

$$\begin{aligned}\frac{d\boldsymbol{t}}{ds} &= \frac{1}{|\boldsymbol{\alpha}'|}\boldsymbol{t}' \\ &= \frac{1}{2\sin(u/2)}\,\frac{1}{2}\left(\cos\frac{u}{2}, -\sin\frac{u}{2}\right) \\ &= -\frac{1}{4\sin(u/2)}\boldsymbol{n}.\end{aligned}$$

The curvature, for $0 < u < 2\pi$, is therefore given by

$$\kappa = -\frac{1}{4\sin(u/2)}\,, \quad 0 < u < 2\pi\ .$$

In fact, for all values of the parameter u,

$$\kappa = -\frac{1}{4|\sin(u/2)|}\ .$$

Notice that the minimum of the absolute value $|\kappa|$ of the curvature for $0 < u < 2\pi$ is $1/4$ at $u = \pi$, and that the curvature approaches $-\infty$ as u approaches 0 and 2π. Indeed, the absolute value of the curvature decreases from ∞ to $1/4$ as u increases from 0 to π and then increases from $1/4$ to ∞ as u increases from π to 2π. This can be seen in the diagram of the curve in Figure 1.4, as can the clockwise direction of rotation of the unit tangent vector $\boldsymbol{t}$ (which is why the curvature is negative).

Now that we have obtained the moving frame equations and given examples of their use, in the remainder of this section we give a geometrical interpretation of the curvature κ, and then state and prove a basic existence and uniqueness theorem for regular curves in the plane.

As may be seen from (1.6), the curvature κ is a measure of how quickly the trace of the curve is bending away from its tangent line when the trace is traversed at unit speed. This is reflected in the following result.

Lemma 3 *The curvature κ of a regular plane curve $\boldsymbol{\alpha}$ is identically zero if and only if $\boldsymbol{\alpha}$ is a straight line.*

Proof If $\kappa = 0$ at each point of $\boldsymbol{\alpha}$ then (1.6) shows that $\boldsymbol{t} = \boldsymbol{c}$, a constant unit vector. In this case, $d\boldsymbol{\alpha}/ds = \boldsymbol{c}$, so $\boldsymbol{\alpha}(s) = \boldsymbol{b} + s\boldsymbol{c}$, for some constant vector $\boldsymbol{b}$. Thus $\boldsymbol{\alpha}$ is the straight line through $\boldsymbol{b}$ in direction $\boldsymbol{c}$. Conversely, a line may be parametrised by arc length as $\boldsymbol{\alpha}(s) = \boldsymbol{b} + s\boldsymbol{c}$, where $\boldsymbol{b}$ is a point on the line and $\boldsymbol{c}$ is a unit vector in the direction of the line. That $\kappa = 0$ at each point of $\boldsymbol{\alpha}$ is now easily checked. □

As mentioned earlier, we may interpret κ as the rate of rotation in the anticlockwise direction of the unit tangent vector $\boldsymbol{t}$, or equivalently of the unit normal vector $\boldsymbol{n}$, as we travel along the curve at unit speed. Here is the proof.

Lemma 4 *Let $\boldsymbol{e}_1, \boldsymbol{e}_2$ denote the standard basis vectors* $(1,0), (0,1)$ *respectively in* $\mathbb{R}^2$. *If θ is the angle from $\boldsymbol{e}_1$ to $\boldsymbol{t}$ measured in an anticlockwise direction (or equivalently, the angle from $\boldsymbol{e}_2$ to $\boldsymbol{n}$), then*

$$\kappa = \frac{d\theta}{ds}\ .$$

Proof The unit tangent vector $\boldsymbol{t}$ is given by (Figure 1.6)

$$\boldsymbol{t} = (\cos\theta, \sin\theta)\,, \quad s \in I\,,$$