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James Carlson , Stefan Müller-Stach , Chris Peters
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Part ONE

BASIC THEORY

1

Introductory Examples

The basic idea of Hodge theory is that the cohomology of an algebraic variety has more structure than one sees when viewing the same object as a "bare" topological space. This extra structure helps us understand the geometry of the underlying variety, and it is also an interesting object of study in its own right. Because of the technical complexity of the subject, in this chapter, we look at some motivating examples which illuminate and guide our study of the complete theory. We shall be able to understand, in terms of specific and historically important examples, the notions of Hodge structure, period map, and period domain. We begin with elliptic curves, which are the simplest interesting Riemann surfaces.

1.1 Elliptic Curves

The simplest algebraic variety is the Riemann sphere, the complex projective space $\mathbf{P}^1$. The next simplest examples are the branched double covers of the Riemann sphere, given in affine coordinates by the equation

$$y^2 = p(x),$$

where $p(x)$ is a polynomial of degree d. If the roots of p are distinct, which we assume they are for now, the double cover C is a one-dimensional complex manifold, or a Riemann surface. As a differentiable manifold it is characterized by its genus. To compute the genus, consider two cases. If d is even, all the branch points are in the complex plane, and if d is odd, there is one branch point at infinity. Thus the genus g of such a branched cover C is $d/2$ when d is even and $(d-1)/2$ when d is odd. These facts follow from Hurwitz's formula, which in turn follows from a computation of Euler characteristics (see Problem 1.1.2). Riemann surfaces of genus 0, 1, and 2 are illustrated in Fig. 1.1. Note

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that if $d = 1$ or $d = 2$, then C is topologically a sphere. It is not hard to prove that it is also isomorphic to the Riemann sphere as a complex manifold.

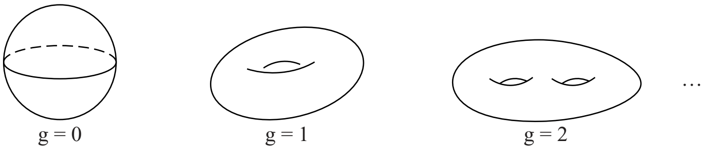

Figure 1.1 Riemann surfaces.

Now consider the case $d = 3$, so that the genus of C is 1. By a suitable change of variables, we may assume the three roots of $p(x)$ to be 0, 1, and λ, where $\lambda \neq 0, 1$:

$$y^2 = x(x-1)(x-\lambda). \tag{1.1}$$

We shall denote the Riemann surface defined by (1.1) by $\mathscr{E}_\lambda$, and we call the resulting family the *Legendre family*. As topological spaces, and even as differentiable manifolds, the various $\mathscr{E}_\lambda$ are all isomorphic, as long as $\lambda \neq 0, 1$, a condition which we assume to be now in force. However, we shall prove the following.

Theorem 1.1.1 *Suppose that $\lambda \neq 0, 1$. Then there is an $\epsilon > 0$ such that for all λ' within distance ϵ from λ, the Riemann surfaces $\mathscr{E}_\lambda$ and $\mathscr{E}_{\lambda'}$ are not isomorphic as complex manifolds.*

Our proof of this result, which guarantees an infinite supply of essentially distinct elliptic curves, will lead us directly to the notions of period map and period domain and to the main ideas of Hodge theory.

The first order of business is to recall some basic notions of Riemann surface theory so as to have a detailed understanding of the topology of $\mathscr{E}_\lambda$, which for now we write simply as $\mathscr{E}$. Consider the multiple-valued holomorphic function

$$y = \sqrt{x(x-1)(x-\lambda)}.$$

On any simply connected open set which does *not* contain the branch points $x = 0, 1, \lambda, \infty$, it has two single-valued determinations. Therefore, we cut the Riemann sphere from 0 to 1 and from λ to infinity, as in Fig. 1.2. Then analytic continuation of y in the complement of the cuts defines a single-valued function. We call its graph a "sheet" of the Riemann surface. Note that analytic continuation of y around δ returns y to its original determination, so δ lies in a single sheet of $\mathscr{E}$. We can view it as lying in the Riemann sphere itself. But when we analytically continue along γ, we pass from one sheet to the other as we pass the branch cut. That path is therefore made of two pieces, one in one sheet and one in the other sheet.

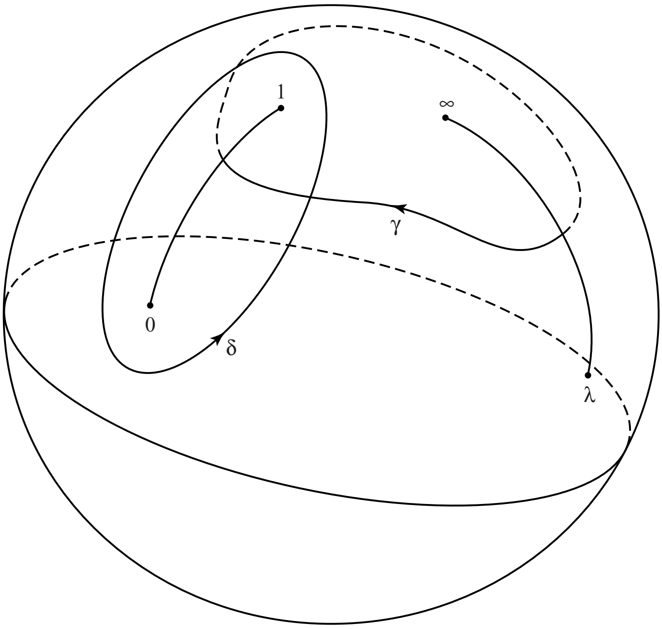

Figure 1.2 Cuts in the Riemann sphere.

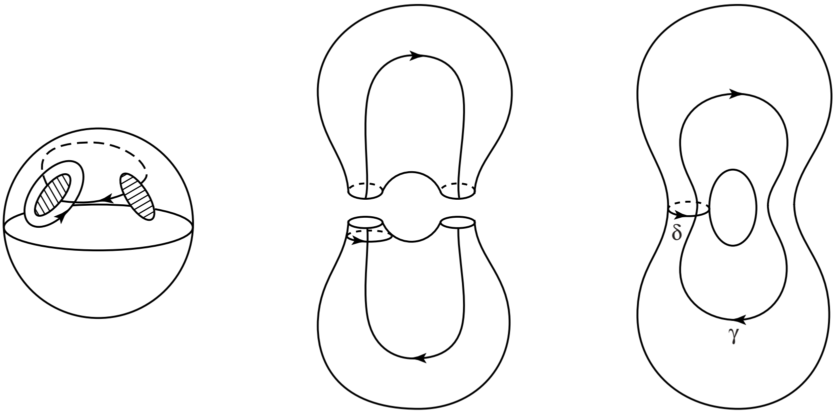

Figure 1.3 Assembling a Riemann surface.

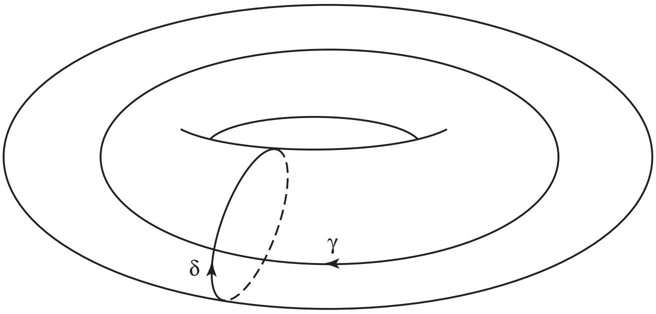

Figure 1.4 Torus.

Thus the Riemann surface of y consists of two copies of the Riemann sphere minus the cuts, which are then "cross-pasted": we glue one copy to the other along the cuts but with opposite orientations. This assembly process is illustrated in Fig. 1.3. The two cuts are opened up into two ovals, the opened-up Riemann sphere is stretched to look like the lower object in the middle, a second copy is set above it to represent the other sheet, and the two sheets are cross-pasted to obtain the final object.

The result of our assembly is shown in Fig. 1.4. The oriented path δ indicated in Fig. 1.4 can be thought of as lying in the Riemann sphere, as in Fig. 1.2, where it encircles one branch cut and is given parametrically by

$$\delta(\theta) = 1/2 + (1/2 + k)e^{i\theta}$$

for some small k. The two cycles δ and γ are oriented oppositely to the x and y axes in the complex plane, and so the intersection number of the two cycles is

$$\delta \cdot \gamma = 1.$$

We can read this information off either Fig. 1.3 or Fig. 1.2. Note that the two cycles form a basis for the first homology of $\mathscr{E}$ and that their intersection matrix is the standard unimodular skew form,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

With this explanation of the homology of our elliptic curve, we turn to the cohomology. Recall that cohomology classes are given by linear functionals on homology classes, and so they are given by integration against a differential form. (This is de Rham's theorem – see Theorem 2.1.1). In order for the line integral to be independent of the path chosen to represent the homology class, the form must be closed. For the elliptic curve $\mathscr{E}$ there is a naturally given differential one-form that plays a central role in the story we are recounting. It is defined by

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}. \tag{1.2}$$

As discussed in Problem 1.1.1, this form is holomorphic, that is, it can be written locally as

$$\omega = f(z)\, dz,$$

where z is a local coordinate and $f(z)$ is a holomorphic function. In fact, away from the branch points, x is a local coordinate, so this representation follows from the fact that $y(x)$ has single-valued holomorphic determinations. Because f is holomorphic, ω is closed (see Problem 1.1.7). Thus it has a well-defined cohomology class.

Now let δ^* and γ^* denote the basis for $H^1(\mathscr{E};\mathbf{Z})$ which is dual to the given basis of $H_1(\mathscr{E};\mathbf{Z})$. The cohomology class of ω can be written in terms of this basis as

$$[\omega] = \delta^* \int_\delta \omega + \gamma^* \int_\gamma \omega.$$

In other words, the coordinates of $[\omega]$ with respect to this basis are given by the indicated integrals. These are called the *periods* of ω. In the case at hand, they are sometimes denoted A and B, so that

$$[\omega] = A\delta^* + B\gamma^*. \tag{1.3}$$

The expression (A, B) is called the *period vector* of $\mathscr{E}$.

From the periods of ω we are going to construct an invariant that can detect changes in the complex structure of $\mathscr{E}$. In the best of all possible worlds this invariant would have different values for elliptic curves that have different complex structures. The first step toward constructing it is to prove the following.

Theorem 1.1.2 *Let $H^{1,0}$ be the subspace of $H^1(\mathscr{E};\mathbf{C})$ spanned by ω, and let $H^{0,1}$ be the complex conjugate of this subspace. Then*

$$\boxed{H^1(\mathscr{E};\mathbf{C}) = H^{1,0} \oplus H^{0,1}.}$$

The decomposition asserted by this theorem is the *Hodge decomposition* and it is fundamental to all that follows. Now there is no difficulty in defining the $(1,0)$ and $(0,1)$ subspaces of cohomology: indeed, we have alreaddy done this. The difficulty is in showing that the defined subspaces span the cohomology, and that (equivalently) their intersection is zero. In the case of elliptic curves, however, there is a quite elementary proof of this fact. Take the cup product of (1.3) with its conjugate to obtain

$$[\omega] \cup [\bar{\omega}] = (A\bar{B} - B\bar{A})\,\delta^* \cup \gamma^*.$$

Multiply the previous relation by $\mathrm{i} = \sqrt{-1}$ and use the fact that $\delta^* \cup \gamma^*$ is the fundamental class of $\mathscr{E}$ to rewrite the preceding equation as

$$\mathrm{i} \int_{\mathscr{E}} \omega \wedge \bar{\omega} = 2\,\mathrm{Im}(B\bar{A}).$$

Now consider the integral above. Because the form ω is given locally by $f\ \mathrm{d}z$, the integrand is locally given by

$$\mathrm{i}|f|^2\ \mathrm{d}z \wedge \mathrm{d}\bar{z} = 2|f|^2\ \mathrm{d}x \wedge \mathrm{d}y,$$

where $\mathrm{d}x \wedge \mathrm{d}y$ is the natural orientation defined by the holomorphic coordinate, that is, by the complex structure. Thus the integrand is locally a positive function times the volume element, and so the integral is positive. We conclude that

$$\mathrm{Im}(B\bar{A}) > 0.$$

We also conclude that neither A nor B can be 0 and, therefore, that the cohomology class of ω cannot be 0. Consequently the subspace $H^{1,0}(\mathscr{E})$ is nonzero.

Because neither A nor B can be 0 we can rescale ω and assume that $A = 1$. For such "normalized" differentials, we conclude that the imaginary part of the normalized B-period is positive:

$$\mathrm{Im}\,B > 0. \tag{1.4}$$

Now suppose that $H^{1,0}$ and $H^{0,1}$ do not give a direct sum decomposition of $H^1(\mathscr{E}; \mathbf{C})$. Then $H^{1,0} = H^{0,1}$, and so $[\bar{\omega}] = \lambda[\omega]$ for some complex number λ. Therefore

$$\delta^* + \bar{B}\gamma^* = \lambda(\delta^* + B\gamma^*).$$

Comparing coefficients, we find that $\lambda = 1$ and then that $B = \bar{B}$, in contradiction with the fact that B has a positive imaginary part. This completes the proof of the Hodge theorem for elliptic curves, Theorem 1.1.2.

An Invariant of Framed Elliptic Curves

Now suppose that $f : \mathscr{E}_\mu \longrightarrow \mathscr{E}_\lambda$ is an isomorphism of complex manifolds. Let ω_μ and ω_λ be the given holomorphic forms. Then we claim that

$$f^*\omega_\lambda = c\omega_\mu \tag{1.5}$$

for some nonzero complex number c. This equation is certainly true on the level of cohomology classes, although we do not yet know that c is nonzero. However, on the one hand,

$$\int_{[\mathscr{E}_\mu]} f^*\omega_\lambda \wedge f^*\bar{\omega}_\lambda = |c|^2 \int_{[\mathscr{E}_\mu]} \omega_\mu \wedge \bar{\omega}_\mu,$$

and on the other,

$$\int_{[\mathscr{E}_\mu]} f^*\omega_\lambda \wedge f^*\bar{\omega}_\lambda = \int_{f_*[\mathscr{E}_\mu]} \omega_\lambda \wedge \bar{\omega}_\lambda = \int_{[\mathscr{E}_\lambda]} \omega_\lambda \wedge \bar{\omega}_\lambda.$$

The last equality uses the fact that an isomorphism of complex manifolds is a degree-one map. Because $\mathrm{i}\omega_\lambda \wedge \bar{\omega}_\lambda$ is a positive multiple of the volume form, the integral is positive and therefore

$$c \neq 0. \tag{1.6}$$

We can now give a preliminary version of the invariant alluded to above. It is the ratio of periods B/A, which we write more formally as

$$\tau(\mathscr{E}, \delta, \gamma) = \frac{\int_\gamma \omega}{\int_\delta \omega}.$$

From Eq. (1.4) we know that τ has a positive imaginary part. From the just-proved proportionality results (1.5) and (1.6), we conclude the following.

Theorem 1.1.3 *If $f : \mathscr{E} \longrightarrow \mathscr{E}'$ is an isomorphism of complex manifolds, then $\tau(\mathscr{E}, \delta, \gamma) = \tau(\mathscr{E}', \delta', \gamma')$, where $\delta' = f_*\delta$ and $\gamma' = f_*\gamma$.*

To interpret this result, let us define a *framed elliptic curve* $(\mathscr{E}, \delta, \gamma)$ to consist of an elliptic curve and an integral basis for the first homology such that $\delta \cdot \gamma = 1$. Then we can say that “if framed elliptic curves are isomorphic, then their τ-invariants are the same.”

Example 1.1.4 In the Legendre family, consider the fiber for $\lambda = -1$, the elliptic curve E. From its equation,

$$y^2 = x^3 - x,$$

we see that the map $(x, y) \mapsto (-x, \mathrm{i}y)$ is an automorphism of this curve of order 4. By Chowla and Selberg (1949), one can explicitly calculate its periods:

$$\int_\gamma \omega = \frac{\Gamma(1/4)^2}{\sqrt{2\pi}}, \quad \int_\delta \omega = -\mathrm{i} \int_\gamma \omega,$$

so that $\tau = \mathrm{i}$. In other words, $E = \mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\mathrm{i}$ and we see that the lattice defining E admits multiplication by i, an extra isomorphism of order 4. In

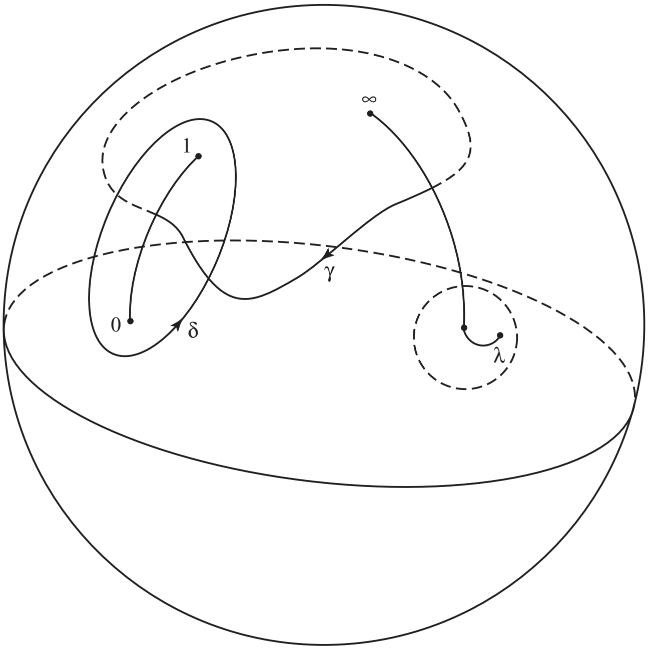

Figure 1.5 Modified cuts in the Riemann sphere.

fact, the lattice is stable under multiplication by the Gaussian integers $\mathbf{Z}[\mathrm{i}] = \{m + \mathrm{i}n \mid n, m \in \mathbf{Z}\}$. We say that E admits *complex multiplication by* $\mathbf{Z}[\mathrm{i}]$.

Holomorphicity of the Period Mapping

Consider once again the Legendre family (1.1) and choose a complex number $a \neq 0, 1$ and an $\epsilon > 0$ which is smaller than both the distance from a to 0 and the distance from a to 1. Then the Legendre family, restricted to λ in the disk of radius ϵ centered at a, is trivial as a family of differentiable manifolds. This means that it is possible to choose two families of integral homology cycles δ_λ and γ_λ on $\mathscr{E}_\lambda$ such that $\delta_\lambda \cdot \gamma_\lambda = 1$. We can "see" these cycles by modifying Fig. 1.2 as indicated in Fig. 1.5. A close look at Fig. 1.5 shows that we can move λ within a small disk Δ without changing either δ_λ or γ_λ. Thus we can view the integrals defining the periods A and B as having constant domains of integration but variable integrands.

Let us study these periods more closely, writing them as

$$A(\lambda) = \int_\delta \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-\lambda)}}, \; B(\lambda) = \int_\gamma \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-\lambda)}}.$$

We have suppressed the subscript on the homology cycles in view of the remarks made at the end of the previous paragraph. The first observation is the following.

Proposition 1.1.5 *On any disk Δ in the complement of the set $\{0, 1, \infty\}$, the periods of the Legendre family are single-valued holomorphic functions of λ.*

The proof is straightforward. Since the domain of integration is constant, we can compute $\partial A/\partial\bar{\lambda}$ by differentiating under the integral sign. But the integrand is a holomorphic expression in λ, and so that derivative is 0. We conclude that the period function $A(\lambda)$ is holomorphic, and the same argument applies to $B(\lambda)$.

Notice that the definitions of the period functions A and B on a disk Δ depend on the choice of a symplectic homology basis $\{\delta, \gamma\}$. Each choice of basis gives a different determination of the periods. However, if δ' and γ' give a different basis, then

$$\delta' = a\delta + b\gamma,$$
$$\gamma' = c\delta + d\gamma,$$

where the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has determinant 1. The periods with respect to the new basis are related to those with respect to the old one as follows:

$$A' = aA + bB,$$
$$B' = cA + dB.$$

Thus the new period vector (A', B') is the product of the matrix T and the old period vector (A, B). The τ-invariants are related by the corresponding fractional linear transformation:

$$\tau' = \frac{d\tau + c}{b\tau + a}.$$

The ambiguity in the definition of the periods and of the τ-invariant is due to the ambiguity in the choice of a homology basis. Now consider a simply connected open set U of $\mathbf{P}^1 - \{0, 1, \infty\}$ and a point λ_0 and λ of U. The choice of homology basis for $\mathscr{E}_{\lambda_0}$ determines a choice of homology basis for all other fibers $\mathscr{E}_\lambda$.