Introduction

Analysis is concerned with continuity and convergence. Investigation of these ideas led to the notions of topology and topological spaces. Once these had been introduced, they became subjects in their own right, which were investigated in fine detail to see how far the theory might lead (an excellent illustration of this is given by the fascinating book by Steen and Seebach [SS]).

In practice, however, a great deal of analysis is concerned with what happens on a very restricted class of topological spaces, namely, the Polish spaces. A Polish space is a separable topological space whose topology is defined by a complete metric. Important examples include Euclidean space, pathwise-connected Riemannian manifolds, compact metric spaces and separable Banach spaces.

The purpose of this book is to develop the study of analysis on Polish spaces. It consists of three parts. The first considers topological properties of Polish spaces, and the second deals with the theory of measures on Polish spaces. In the third part, we give an introduction to the theory of optimal transportation. This makes essential use of the results of the first two parts, or modifications of them. It was, in fact, study of optimal transportation that led to the realization of how much its study required properties of Polish spaces, and measures on them.

There are three important advantages of restricting attention to Polish spaces. First, many of the curious complications of the general topological theory disappear. For example, a subspace of a separable topological space need not be separable, whereas a subspace of a separable metric space is always separable. Secondly, the proofs of standard results are frequently much easier in this restricted setting. For example, Urysohn's lemma for normal topological spaces is quite delicate, whereas it is very easy for metric spaces. Thirdly, Polish spaces enjoy some very important properties. Thus it follows from Alexandroff's theorem that a topological space is a Polish space if and only if it is homeomorphic to a G_δ subset of the Hilbert cube $\mathbf{H} = [0, 1]^{\mathbf{N}}$, which is a compact metrizable space. From this, or directly, it follows that a Borel measure on a Polish space is tight (Ulam's theorem: the measure of a Borel set can be approximated from below by the measures of compact sets contained in it). It also means that we can push forward a Borel measure on a Polish space X to a Borel measure on a compact metric space containing X. This greatly simplifies both the measure theory and also the construction of measures. In fact, I believe that almost all the probability measures that arise in practice are Borel measures on Polish spaces; one important exception, which we do not consider or need, is the theory of uniform central limit theorems.

One major advantage of restricting attention to Polish spaces is that it is not necessary to appeal to the axiom of choice. Instead, we proceed by induction, using the axiom of dependent choice; we make an infinite sequence of decisions, each possibly dependent on what has gone before.

In analysis, there are a few fundamental results which require the axiom of choice. The first is Tychonoff's theorem, which states that an arbitrary product of compact topological spaces, with the product topology, is compact. We do not prove this, or use it. On the other hand, we do prove, and use, the fact that a countable product of compact metrizable spaces is compact and metrizable.

Secondly, there are two fundamental results of linear analysis which need the axiom of choice, using Zorn's lemma. The first of these is the Hahn–Banach theorem (together with the separation theorem). Using induction, we prove weak forms of these, for separable normed spaces; this is sufficient for our purposes.

But for completeness' sake we also give the classical results, using Zorn's lemma; Here we first prove the separation theorem, showing that it essentially depends upon the connectedness of the unit circle $\mathbf{T}$, and then derive the Hahn–Banach theorem from it.

The other fundamental result which requires the axiom of choice is the Krein–Mil'man theorem, which states that every weakly compact convex subset K has an extreme point. Again, we only need, and use, the result in the case where K is metrizable, and we prove this without the axiom of choice.

The fact that we avoid using the axiom of choice suggests that the proofs should, in some sense, be less abstract and more constructive. Unfortunately, this is not the case; the arguments that are used are frequently indirect (consider the collection of all sets with a particular property), so that for example a typical Borel subset of a Polish space does not have a simple description.

Let us now describe the contents of the three parts of this book in more detail.

Part I: Topological Properties

Although it is assumed that the reader has some knowledge of general topology and metric spaces, the first two chapters give an account of these topics, including Tietze's extension theorem, Baire's category theorem and Lipschitz functions.

This leads to the notion of a Polish space, a separable topological space whose topology is given by a complete metric. A fundamental example is given by a compact metrizable space, and Alexandroff's theorem is used to show that a topological space is a Polish space if and only if it is homeomorphic to a G_δ subspace of a compact metric space, and in particular homeomorphic to a G_δ subspace of the Hilbert cube.

We shall need to consider suprema of sets of real-valued continuous functions. Such functions are lower semi-continuous, and we consider such functions in Chapter 4. A lower semi-continuous function on a compact space attains its infimum, but this is not necessarily true for lower semi-continuous functions on a complete metric space. We establish its replacement, Ekeland's variational principle, together with two of its corollaries, the petal theorem and Daneš's drop theorem, and various other applications.

Metric spaces have more structure than a topological one, and Chapter 5 contains an account of uniform spaces; uniformity is particularly important when we consider locally compact topological groups, in Part II.

Chapter 6 is devoted to showing that the space of càdlàg functions is a Polish space under the Skorohod topology; many stochastic processes, and their underlying measures, lie on such spaces, and this helps justify the claim that almost all probability measures of interest lie on Polish spaces. Further examples are given by separable Banach spaces and Hilbert spaces; these are principally used to introduce the notion of convexity.

The rest of Part I is concerned with convexity. The Hahn–Banach theorem is one of the key results here, and we give proofs of appropriate results, both without and with the axiom of choice. For us, the Hahn–Banach theorem is essentially a geometric theorem showing that two suitable convex sets can be separated by a hyperplane. It also leads onto the notion of weak topology.

The Legendre transform provides an important duality theory for convex functions, and this leads naturally to the concept of subdifferentials and subdifferentiability. We prove the Bishop–Phelps theorem, and also introduce the notion of cyclic monotonicity.

The rest of Part I is concerned with convex sets which are compact and metrizable in some suitable topology. We prove versions of the Krein–Mil'man theorem, Krein's theorem and a swathe of fixed point theorems, many of which are used later.

Part II: Measures on Polish Spaces

We expect that the reader has some knowledge of abstract measure theory, but Chapter 14 contains a survey of the basic results. Chapter 15 contains some further results: we introduce the Banach space $M(X)$ of finite measures on a Polish space X, its subspaces $L^1(\mu)$ and Orlicz spaces (with the use of Legendre duality). We give von Neumann's Hilbert space proof of the Radon–Nikodym property and a proof of the strong law of large numbers (to be used later).

In Chapter 16, we investigate Borel measures on Polish spaces. We prove regularity and tightness properties; we may not know what a typical Borel set looks like, but we can approximate the Borel measure of a Borel set from the outside by open sets, and on the inside by compact sets. This leads to Lusin's theorem, which says that if μ is a Borel measure on a Polish space X then a Borel measurable function on X is continuous on a large compact subset.

So far, all is theory, and no measures, other than trivial ones, have been shown to exist. We remedy this by showing how to construct Borel measures on the Bernoulli space $\Omega(\mathbf{N})$, and then, pushing forward, constructing measures on compact metric spaces and Polish spaces. We prove the Riesz representation theorem, and use this to give a measure-theoretic proof of the Stone–Weierstrass theorem.

We then show how Borel measures can be disintegrated, and establish the existence of Haar measure on compact and locally compact Polish spaces; we follow an account by Pedersen to show that this last result is relatively straightforward.

In Chapter 17, we come down to earth and consider Borel measures on Euclidean space, where the point at issue is the differentiation of measures and of Borel measurable functions. We establish Lebesgue's differentiation theorem and Rademacher's theorem on the differentiability almost everywhere of Lipschitz functions.

We now proceed to study one of the key points of this chapter, namely, the weak convergence of measures. We show that there are various metrics which define the weak topology w, and show that although the unit ball $M_1(X)$ is generally not metrizable, the space of probability measures $P(X)$ is a Polish space. Examples of weak convergence include the central limit theorem and the empirical law of large numbers. Finally, uniform integrability is investigated.

Part II ends with an introduction to Choquet theory on a metrizable compact convex set. The theory is notoriously difficult for general weakly compact convex sets, but the difficulties disappear in the metrizable case.

Parts I and II contain more than two hundred exercises. These are usually very straightforward, but most are an essential part of the text; do them.

Part III: Introduction to Optimal Transportation

The setting is this; μ and ν are Borel probability measures on Polish spaces X and Y, and c is a lower semi-continuous cost function on $X \times Y$. We consider two problems. Kantorovich's problem is to find a measure π on $X \times Y$ with marginals μ and ν with minimal cost $\int_{X\times Y} c\,d\pi$. Monge's problem is a special case of this; find a measurable mapping $T : X \to Y$ which pushes forward μ to ν with minimal cost $\int_X c(x, T(x))\,d\mu(x)$. The results of Parts I and II are used, or modified, to tackle these problems. For example we can push forward μ and ν to measures on metrizable compactifications. We also consider the concepts of c-cyclic monotonicity and c-concavity. It is quite easy to show that Kantorovich's problem has a solution, but with more care we introduce a 'maximal Kantorovich potential', which with its c-transform can give a great deal of information.

When $X = Y$ and $c = d^p$, where d is a metric on X, we introduce and investigate the Wasserstein metric W_p, which is the minimal cost of transforming μ into ν. Similarly, we introduce the Mallows distance, which does the same for distributions of random variables. As an example, we prove a metric version of the central limit theorem.

In the last chapter, we consider special cases. For example, we consider the case when $X = Y = \mathbf{R}$, and the case where the cost is a quadratic function on a separable Hilbert space. Finally, following Gangbo and McCann [GMcC], we consider the cases when the cost on $\mathbf{R}^d$ is given by a strictly convex or strictly concave function.

This only scratches the surface: for more, see the two large volumes by Villani, [V I] and [V II].

Although I have checked the proofs carefully, no doubt errors remain. Please consult my home page at **www.dpmms.cam.ac.uk** where a list of comments and corrections will be found, together with my email address, to which corrections should be sent.