1

Introduction

Ich behaupte aber, dass in jeder besonderen Naturlehre nur so viel eigentliche
Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen ist.
[But I maintain that in every special natural doctrine only so much science proper
is to be met with as mathematics.]

Immanuel Kant, 1786

Eratosthenes of Cyrene, who lived basically in the third century BC, was one of those first
mathematicians whose knowledge and abilities at these early stages of human civilization
were remarkable. Besides his method for seeking prime numbers, he particularly also con-
tributed to the measurement of the Earth by, for example, determining its circumference.
In this respect, he might have been the first geomathematician, or at least one of the first.
Many more followed him, and definitely Carl Friedrich Gauß must be mentioned here, as
he can be seen as the greatest genius in mathematical history. His works and their influ-
ence are widespread in mathematics, and they are also of essential importance in various
applications, in particular and (in the author’s possibly biased point of view) first of all in
Earth sciences, especially geomagnetics and potential theory. The awareness, which reaches
back to classical antiquity, that mathematics is the foremostly required skill and toolbox
for understanding the objects and processes that surround us has been preserved up to the
present. It has nicely and more generally been put in a nutshell by the preceding quotation,
which is from Kant (1786); for the English translation, see Kant (1883). Over the centuries,
Earth sciences and mathematics have both advanced. While the achievements at the time of
Eratosthenes and his fellows are nowadays parts of the curricula at schools, many modern
challenges in geosciences are equally challenges to twenty-first century mathematics.

The importance of mathematics for the understanding of the entire phenomenologies
which are associated to the Earth was recognized and highlighted by the initiative Mathe-
matics of Planet Earth (MPE), which was launched by UNESCO in 2013. Since then, the
interest in geomathematics has grown extensively and many publications have occurred in
the wake of MPE. Certainly, the Earth is as complex as it is versatile. Therefore, one single
book cannot cover the whole mathematics which models processes occurring in the Earth, at
its surface, and in the atmosphere. This new book that you are currently reading concentrates
on specific topics: gravitation, magnetics, and seismology, though also these fields could
easily fill books of the same size on their own. Moreover, this book is not only devoted
to the modelling of these physical areas, but also to the mathematical foundations which
are necessary for understanding and solving the occurring challenges. This comprises, in particular, a selection of basis function systems, including an algorithm for best basis choice, and the theory of inverse problems and their regularization.

Given the limitations that naturally occur when an author tries to squeeze these topics into approximately 500 pages, one needs to concentrate on certain essentials. A particular focus was put on the interconnections. There are mathematical tools which are essential in more than one of the three considered Earth sciences. Nevertheless, each chapter can be read on its own, but cross references are set where theorems and concepts are needed which have been derived in other contexts. Looking back at approximately a quarter of a century of being a scientist, the author has noticed that often the solution of one’s currently urgent problems is located beyond one’s own nose. Moreover, for opening the right drawer, one needs to know what is inside each drawer – otherwise the search can become very time consuming. Therefore, gravity experts, geomagnetics scientists, and seismologists are particularly encouraged to look not only into those chapters which have headings that are familiar in their own disciplines. Another focus was set on clear and rigorous deductions. For example, in the chapter on gravitation, we start with nothing more than Sir Isaac Newton’s law of gravitation, which leads us to an integral. Then we observe the properties of this integral and their consequences. This brings us automatically, for instance, to the modelling with spherical harmonics and radial basis functions. Many tools and common facts, which are often accepted without scrutinizing them, occur here as logical consequences of elementary starting points in the modelling.

This book addresses several groups of readers: mathematicians who are interested in the theoretical foundations of certain areas of Earth sciences but also those who need tools for solving applied problems with mathematical means; geoscientists who look for a mathematical reference which explains why common physical realities are actually mathematical facts; and also those who look for new inspirations and alternative approaches for solving various topical problems in their fields. The monograph has been written for students as well as researchers who will both find their own particular benefit from it. It was also very important for the author to produce a book where proofs and other derivations are comprehensible to a wide audience. Numerous graphical illustrations support the understanding of complicated mathematical concepts. Moreover, for finding important keywords more easily, many of them have been set in boldface.

Acknowledgements: Last but not least, the author, who switches here to the first person, feels the strong need to thank many people who made this book possible and who helped to bring it to what it has become. The largest gratitude is owed Kornelia Mielke, who coped with my handwriting and translated it into \LaTeX{} for major parts of this book. Without her, the book would be far from finished yet. Certainly, I am also grateful to the people at Cambridge University Press for giving me the opportunity to publish my book under their famous brand and for being extremely patient with me, because I should have actually finished this work years ago.

Moreover, I am beholden to those members of my group and students who have been proofreading various intermediate versions of this book. In particular, I want to mention
Amna Ishtiaq, Max Kontak, Bianca Kretz, Sarah Leweke, Naomi Schneider, and Katrin Seibert. The latter two also earn my gratitude regarding some of the numerical calculations and the corresponding plots. Furthermore, there are also some other people without whom this book might have never been written: those many students in Kaiserslautern and Siegen who attended some of my master courses and were missing accompanying textbooks, which was the reason why they asked me to fill this gap.

And now definitely last but surely not least: with special emphasis, I want to thank Willi Freeden, my academic teacher, mentor, and friend, for awakening my lifelong enthusiasm for geomathematics.
2
Required Mathematical Basics

2.1 Some Important Definitions

We introduce here some notations and mathematical nomenclatures which we will need frequently within the book. First of all, $\mathbb{R}$ denotes, as usual, the set of all real numbers, whereas $\mathbb{C}$ stands for the set of all complex numbers, $\mathbb{Z}$ for the set of all integers, and $\mathbb{N}$ for the set of all positive integers with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Furthermore, $\mathbb{R}^+$ represents the set of all positive real numbers, where $\mathbb{R}^+_0 := \mathbb{R}^+ \cup \{0\}$, etc.

**Definition 2.1.1** Let $x^0 \in \mathbb{R}^n$ and $R \in \mathbb{R}^+$ be given.

(a) The (open) **ball** with centre $x^0$ and radius $R$ is defined by $B_R(x^0) := \{ x \in \mathbb{R}^n \mid |x - x^0| < R \}$ such that $\overline{B}_R(x^0) := \{ x \in \mathbb{R}^n \mid |x - x^0| \leq R \}$ is the closed ball.

(b) The corresponding **sphere** is $S_R(x^0) := \{ x \in \mathbb{R}^n \mid |x - x^0| = R \}$.

(c) The **unit sphere** in $\mathbb{R}^3$ is denoted by $\Omega := S_1(0)$.

Here, $| \cdot |$ represents the **Euclidean norm** with $|x| = \sqrt{\sum_{j=1}^n x_j^2}, x \in \mathbb{R}^n$.

The following definition is based on Walter (1990).

**Definition 2.1.2** Let $R \subset \mathbb{R}^n$ and $x_1, \ldots, x_k \in \mathbb{R}^n, k \in \mathbb{N}$.

(a) We call $x_1x_2 := \{(1-t)x_1 + tx_2 \mid 0 \leq t \leq 1\}$ a **line segment**.

(b) The **polygonal chain** $x_1x_2 \cdots x_k$ is the composition of the line segments through $x_1, x_2, \ldots, x_k$, that is, $x_1x_2 \cdots x_k := x_1x_2 \cup x_2x_3 \cup \cdots \cup x_{k-1}x_k$.

(c) The set $R$ is called **connected**, if every arbitrary pair of points $x, y \in R$ can be connected by a polygonal chain which remains in $R$, that is, there exist $z_1, \ldots, z_j \in R$ (for some $j \in \mathbb{N}$) such that $xz_1 \cdots z_jy \subset R$; see also Figure 2.1.

(d) The set $R$ is called **open**, if the following holds true: for every $x \in R$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset R$.

(e) The set $R$ is called a **region**, if $R$ is non-empty, open, and connected.

(f) The set $R$ is called **bounded**, if there exists $r > 0$ such that $R \subset B_r(0)$.

(g) A point $x \in \mathbb{R}^n$ is called an **accumulation point** of $R$, if, for every $\varepsilon > 0$, the intersection $(B_\varepsilon(x) \setminus \{x\}) \cap R$ is non-void.

(h) The set $R$ is called **closed**, if $R$ contains all of its accumulation points.

(i) The set $R$ is called **compact**, if it is closed and bounded.
2.1 Some Important Definitions

Figure 2.1 The light grey set (left) is not connected. For example, the two black points cannot be connected by a polygonal chain which remains in the set. The dark grey set (right) is connected – if it is also open (i.e., it does not contain its boundary), then it is a region. Any choice of two points in the set can be connected by a polygonal chain within the set.

Figure 2.2 The light grey set (left) is a union of two balls. The distance of their two centres is less than the sum of the two radii. Hence, this set is connected. In the case of the dark grey set (right), the distance is larger than the sum of the two radii and the set is not connected.

For example, every ball $B_R(x_0)$ is a region (for $x,y \in B_R(x_0)$, we always have $\overline{x_0y} \subset B_R(x_0)$). However, the union of two balls $B_{R_0}(x_0) \cup B_{R_1}(x_1)$ is not a region, if $|x_0 - x_1| > R_0 + R_1$, since there is, for example, no polygonal chain connecting $x_0$ and $x_1$ (see Figure 2.2 for an illustration).

Besides the Euclidean norm, there also exist other commonly used operations for vectors in $\mathbb{R}^n$.

**Definition 2.1.3** Let $x, y \in \mathbb{R}^n$. Then $x \cdot y := \langle x, y \rangle_{\mathbb{R}^n} := \sum_{j=1}^n x_j y_j$ represents the Euclidean inner product, the Euclidean scalar product, or the dot product of $x$ and $y$. Vectors $x_1, \ldots, x_k \in \mathbb{R}^n$ are called orthogonal, if $x_i \cdot x_j = 0$ for all $i, j$ with $i \neq j$. They are called orthonormal if they are orthogonal and additionally satisfy $|x_i| = 1$ for all $i$. Note that $|x| = \sqrt{x \cdot x}$ for all $x \in \mathbb{R}^n$. Moreover, in the case $n = 3$,

$$x \times y := \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} \in \mathbb{R}^3$$

is called the cross product or the vector product of $x$ and $y$.

Furthermore, the tensor product (dyadic product) of two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ is the $n \times m$-matrix $x \otimes y := (x_j y_k)_{j=1, \ldots, n, k=1, \ldots, m}$.

Eventually, for vectors $w, z \in \mathbb{C}^n$, the common inner product is defined by $w \cdot z := \langle w, z \rangle_{\mathbb{C}^n} := \sum_{j=1}^n w_j \overline{z_j}$, where $\overline{z_j}$ is the complex conjugation of $z_j$.

**Theorem 2.1.4** The vector product can be represented by using the Levi-Civitè (alternating) symbol
Required Mathematical Basics

\( \varepsilon_{ijk} := \begin{cases} 
1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\
-1, & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\
0, & \text{else}
\end{cases} \)

as \( x \cdot y = \left( \sum_{j,k=1}^{3} \varepsilon_{ijk} x_j y_k \right)_{i=1,2,3} \). Moreover, the following identities hold true:

\[ \sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}, \]

\[ \sum_{i,j,k=1}^{3} (\varepsilon_{ijk})^2 = 6, \]

where \( \delta_{ij} \) with \( \delta_{ij} := 1 \) for \( i = j \) and \( \delta_{ij} := 0 \) for \( i \neq j \) is the Kronecker delta.

This theorem is easy to verify such that we omit the proof here.

We summarize here a few basic properties of the vector product, which can be easily proved by using the Levi-Civita symbol.

**Theorem 2.1.5** Let \( a, b, c, d \in \mathbb{R}^3 \). Then \( a \times b = -b \times a \) and

\[ (a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b. \]

Moreover, we have the expansion theorem

\[ a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \]

and the Lagrange identity

\[ (a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d). \]

**Definition 2.1.6** The standard orthonormal basis in \( \mathbb{R}^3 \) is denoted by \( e_1 := (1,0,0)^T \), \( e_2 := (0,1,0)^T \), and \( e_3 := (0,0,1)^T \). Note that the notations \( e^1, e^2, e^3 \) are also common in the literature.

We will need the well-known Landau symbols.

**Definition 2.1.7** (Landau Symbol) Let \( f \) and \( g \) be two functions (where \( g \) is scalar valued) for which a limit \( x \to a \) with a particular \( a \in \mathbb{R} \cup \{-\infty, +\infty\} \) is declared. We say that \( f(x) = \mathcal{O}(g(x)) \) as \( x \to a \), if \( \frac{f(x)}{g(x)} \) is bounded in a neighbourhood of \( a \) (for \( a \notin \mathbb{R} \), the neighbourhood is \( ]-\infty, b[ \) or \( ]b, +\infty[ \), respectively, for some \( b \in \mathbb{R} \). We say that \( f(x) = o(g(x)) \) as \( x \to a \), if \( \lim_{x \to a} (f(x)/g(x)) = 0 \). Analogously, the Landau symbols can be defined for the left-hand and the right-hand limit, that is, \( x \to a+ \) or \( x \to a- \) for \( a \in \mathbb{R} \). Moreover, limits to vectors \( a \in \mathbb{R}^n \) are also possible.

**Example 2.1.8** Simple examples are as follows: \( \sin x = \mathcal{O}(x) \) as \( x \to 0 \), since \( \lim_{x \to 0} (x^{-1} \sin x) = 1 \). Moreover, \( \sin x = o(x) \) as \( x \to +\infty \) and, consequently, \( \sin x = \mathcal{O}(x) \) as \( x \to +\infty \), since \( \lim_{x \to +\infty} (x^{-1} \sin x) = 0 \).

There are some other concepts that we need.

© in this web service Cambridge University Press  
www.cambridge.org
2.2 A Short Course on Tensors

Definition 2.1.9 Let \( D \subset \mathbb{R}^n \) be an arbitrary set. Then the characteristic function, which is also called the indicator function, \( \chi_D : \mathbb{R}^n \to \mathbb{R} \) corresponding to \( D \) is defined by \( \chi_D(x) := 1 \) for \( x \in D \) and \( \chi_D(x) := 0 \) for \( x \notin D \).

Definition 2.1.10 The Gauß brackets are defined by

\[
[x] := \min \{ n \in \mathbb{Z} \mid x \leq n \}, \quad \lceil x \rceil := \max \{ n \in \mathbb{Z} \mid n \leq x \}
\]

for \( x \in \mathbb{R} \). They represent rounding up and down.

Some of the properties which have been defined in this section can also be defined in more general spaces than \( \mathbb{R}^n \), as we will see in Section 2.5. Moreover, we clarify that, throughout this book, log always stands for the natural logarithm, which is also known as ln.

2.2 A Short Course on Tensors

We will only need some simple aspects of tensors as tools, mainly in the seismological modelling in Chapter 7. The following introduction is based on Dahlen and Tromp (1998, appendix A), where further details also can be found.

Definition 2.2.1 A tensor of order \( q \in \mathbb{N} \setminus \{0\} \) is a functional \( T : (\mathbb{R}^3)^q \to \mathbb{R} \), which is linear in all \( q \) arguments (which are also called the slots of the tensor). This means that

\[
T(\lambda_1 x^{(1)}, \ldots, \lambda_q x^{(q)}) = \left( \prod_{j=1}^{q} \lambda_j \right) T(x^{(1)}, \ldots, x^{(q)})
\]

and

\[
T(x^{(1)}, \ldots, x^{(k-1)}, x^{(k)}, x^{(k+1)}, \ldots, x^{(q)}) = T(x^{(1)}, \ldots, x^{(k-1)}, x^{(k)}, x^{(k+1)}, \ldots, x^{(q)})
\]

\[
+ T(x^{(1)}, \ldots, x^{(k-1)}, y, x^{(k+1)}, \ldots, x^{(q)})
\]

for all \( \lambda_1, \ldots, \lambda_q \in \mathbb{R} \), all \( x^{(1)}, \ldots, x^{(q)}, y \in \mathbb{R}^3 \), and all \( k = 1, \ldots, q \). Furthermore, scalar multiplication and addition of tensors are defined in the usual way for mappings, that is, \( (\lambda_1 T_1 + \lambda_2 T_2)(x) := \lambda_1 T_1(x) + \lambda_2 T_2(x) \) for all \( \lambda_1, \lambda_2 \in \mathbb{R} \) and all \( x \in (\mathbb{R}^3)^q \), if \( T_1 \) and \( T_2 \) are both tensors of the same order \( q \in \mathbb{N} \setminus \{0\} \). Eventually, a tensor of order \( 0 \) is a real number (that is, the slots are considered as non-existent).

The tensor product which we encountered in Definition 2.1.3 is only a particular case of the following definition. Note that we will soon have a closer look on the link between tensors, vectors, and matrices.

Definition 2.2.2 Let \( S \) and \( T \) be tensors of orders \( p \in \mathbb{N} \) and \( q \in \mathbb{N} \), respectively. Then the tensor product or dyadic product \( S \otimes T \) is a tensor of order \( p + q \) which is given by

\[
(S \otimes T)(x^{(1)}, \ldots, x^{(p+q)}) := S(x^{(1)}, \ldots, x^{(p)}) T(x^{(p+1)}, \ldots, x^{(p+q)})
\]

for all \( x^{(1)}, \ldots, x^{(p+q)} \in \mathbb{R}^3 \).
Required Mathematical Basics

Note that the tensor product is not commutative, that is, we have, in general, \( S \otimes T \neq T \otimes S \).

**Definition 2.2.3** Within Section 2.2, \((b^{(1)}, b^{(2)}, b^{(3)})\) denotes an arbitrary choice of an orthonormal basis of \(\mathbb{R}^3\) (with respect to the Euclidean inner product), where the matrix whose columns are built by these vectors has a positive determinant (that is the basis is right-handed). Such a system is called a Cartesian axis system.

An example of such a system is given by the standard orthonormal basis \(e^1, e^2, e^3\); see Definition 2.1.6.

**Definition 2.2.4** Let \(T\) be a tensor of order \(q \in \mathbb{N} \setminus \{0\}\). Then the \(3^q\) components of \(T\) with respect to \((b^{(1)}, b^{(2)}, b^{(3)})\) are defined by

\[
T_{i_1 \ldots i_q} := T \left( (b^{(i_1)}), \ldots, (b^{i_q}) \right)
\]

for all tuples \((i_1, \ldots, i_q) \in \{1, 2, 3\}^q\).

Usually, we choose our Cartesian axis system as one of our first steps when we do any modelling. In this respect, there is a fixed and clearly defined basis which underlies all the considerations. Therefore, tensors are usually represented by their components (since they are one-to-one associated to the tensors, once we have chosen our Cartesian axis system). With this in mind, we can also consider vectors and matrices as particular cases of tensors, as the following example demonstrates.

**Example 2.2.5** Let us have a closer look at the cases of orders 1 or 2.

(a) Let \(c\) be a tensor of order 1. Then \(c\) has three components, namely \(c_1 = c(b^{(1)}), c_2 = c(b^{(2)}), c_3 = c(b^{(3)})\). Remember that every \(x \in \mathbb{R}^3\) can be represented by \(x = \sum_{j=1}^3 (x \cdot b^{(j)})b^{(j)}\), where ‘\(\cdot\)’ is the Euclidean dot product (see Definition 2.1.3). Hence, the linearity of the tensor \(c\) yields

\[
c(x) = c \left( \sum_{j=1}^3 (x \cdot b^{(j)})b^{(j)} \right) = \sum_{j=1}^3 (x \cdot b^{(j)})c(b^{(j)}) = \sum_{j=1}^3 (x \cdot b^{(j)})c_j.
\]

Moreover, after having chosen our Cartesian axis system, we would usually write \(x_j := x \cdot b^{(j)}\). Thus, if we associate the tensor \(c\) to a vector \(c \in \mathbb{R}^3\) with the components \(c = (c_1, c_2, c_3)^T\), then ‘the tensor \(c\) and the vector \(c\)’ satisfy \(c(x) = x \cdot c = x^Tc\). In particular, \(c(b^{(j)}) = b^{(j)} \cdot c = c_j\). Furthermore, every vector can be uniquely associated to a tensor of order 1 (again, provided that the Cartesian axis system has already been chosen).

(b) Let \(A\) be a tensor of order 2, which, consequently, has \(3^2 = 9\) components \(A_{jk} = A(b^{(j)}, b^{(k)})\), \(j, k = 1, 2, 3\). This suggests that \(A\) can be associated to a matrix, where we would usually write \(a_{jk} := A_{jk}\) for the components. If we represent \(x, y \in \mathbb{R}^3\) with respect to the Cartesian axis system \((b^{(1)}, b^{(2)}, b^{(3)})\) as \(x = (x_1, x_2, x_3)^T\) and \(y = (y_1, y_2, y_3)^T\), then we get, due to the bilinearity,
2.2 A Short Course on Tensors

\[ A(x, y) = \sum_{j,k=1}^{3} x_j y_k A(b^{(j)}, b^{(k)}) = \sum_{j,k=1}^{3} x_j y_k a_{j k}. \]

Hence, the tensor \( A \) is the quadratic form associated to the matrix \( A \), that is, \( A(x, y) = x^T Ay \). In analogy to the preceding case, we get here a one-to-one relation between tensors of order 2 and matrices (also here, provided that the Cartesian axis system is fixed).

(c) The tensor product which we know from Definition 2.1.3 is, indeed, a particular case of the tensor product in Definition 2.2.2. If \( x = (x_1, x_2, x_3)^T \) and \( y = (y_1, y_2, y_3)^T \) are arbitrary vectors in \( \mathbb{R}^3 \) (or tensors of order 1), then Definition 2.2.2 defines the tensor product as the tensor of order 2 whose components satisfy

\[ (x \otimes y)_{jk} = (x \otimes y)(b^{(j)}, b^{(k)}) = x(b^{(j)})y(b^{(k)}) = x_jy_k \]

for all \( j, k = 1, 2, 3 \).

Note also that the Levi-Civita alternating symbol, see Theorem 2.1.4, represents components of a tensor of order 3.

**Definition 2.2.6** We define the trace of a tensor and its generalization.

(a) The **trace** of a tensor \( T \) of order 2 (or, in the sense explained previously, a matrix \( T \)) is defined by \( \text{tr\,} T := \sum_{j=1}^{3} T(b^{(j)}, b^{(j)}) = \sum_{j=1}^{3} T_{jj} \). Hence, \( \text{tr\,} T \) is a tensor of order 0, that is, a real number.

(b) The **contraction** of a tensor \( T \) of order \( q \in \mathbb{N} \setminus \{0, 1\} \) upon the \( r \)th and the \( s \)th slot of \( T \) is defined by

\[ \text{tr}_{rs} T := \sum_{j=1}^{3} T \left( \ldots, \frac{b^{(j)}}{\text{slot } r}, \ldots, \frac{b^{(j)}}{\text{slot } s}, \ldots \right), \]

provided that \( r, s \in \{1, \ldots, q\} \) with \( r \neq s \) and ‘\ldots' means that the corresponding slots remain untouched.

Obviously, \( \text{tr}_{rs} T \) is a tensor of order \( q - 2 \) and, if \( T \) is a tensor of order 2, then \( \text{tr}_{12} T = \text{tr\,} T \).

**Definition 2.2.7** We define now the transpose of a tensor, first for the case of order 2, where we recognize the usual transpose of a matrix, and then for general tensors.

(a) The **transpose** of a second-order tensor \( T \) is defined as the second-order tensor \( T^T \), which satisfies

\[ T^T(x, y) := T(y, x) \quad \text{for all } x, y \in \mathbb{R}^3. \]  \hspace{1cm} (2.1)

\( T \) is called **symmetric**, if \( T = T^T \).
Required Mathematical Basics

(b) The transpose of a tensor $T$ of order $q \in \mathbb{N} \setminus \{0, 1\}$ with respect to the $r$th and the $s$th slot is defined via

$$\Pi_{rs} T \left( \begin{array}{c} x \\ \text{slot } r \\ \end{array} \right) \rightarrow \left( \begin{array}{c} y \\ \text{slot } s \\ \end{array} \right) = T \left( \begin{array}{c} y \\ \text{slot } s \\ \end{array} \right) \rightarrow \left( \begin{array}{c} x \\ \text{slot } r \\ \end{array} \right)$$

for all $x, y \in \mathbb{R}^3$, where again ‘...’ stands for untouched slots.

In terms of components, (2.1) means that $T^T_{jk} = T_{kj}$ for all $j, k = 1, 2, 3$.

**Definition 2.2.8** Let $S$ and $T$ be tensors of orders $p \in \mathbb{N}$ and $q \in \mathbb{N}$, respectively, where $q \leq p$. Then the **double-dot product** $S : T$ is the tensor of order $p - q$, which is defined by its components as

$$(S : T)_{j_1, \ldots, j_p} := \sum_{k_1, \ldots, k_q=1}^3 S_{j_1 \ldots j_p, k_1 \ldots k_q} T_{k_1 \ldots k_q}.$$

Note that an analogous definition for complex-valued matrices (or complex tensors of order 2) is given in Example 2.5.2, part c.

**Example 2.2.9** The double-dot product contains particular cases which are known from vector and matrix calculations.

(a) Let $S$ be a tensor of order 2 and $t$ be a tensor of order 1; then the double-dot product yields the tensor of order 1 with $(S : t)_j = \sum_{k=1}^3 S_{jk} t_k$. In other words, if $S$ is a matrix and $t$ is a vector, then the vector $S : t$ is the usual matrix-vector multiplication $St$.

(b) Let $s$ and $t$ be tensors of order 1. Then $s : t$ is the tensor of order 0 which is given by $s : t = \sum_{k=1}^3 s_k t_k$. The analogous terms are here: if $s$ and $t$ are both vectors, then the real number $s : t$ is given by the Euclidean inner product $s \cdot t$.

(c) Let $x$ and $y$ be tensors of order 1 and let $\Lambda$ be the tensor of order 3 whose components are given by the Levi-Civita alternating symbol $\varepsilon_{ijk}$ (see Theorem 2.1.4). Then $\Lambda : (x \otimes y)$ is a tensor of order $3 - (1 + 1) = 1$. Its components satisfy

$$\left( \Lambda : (x \otimes y) \right)_j = \sum_{j, k=1}^3 \varepsilon_{ijk} (x \otimes y)_{jk} = \sum_{j, k=1}^3 \varepsilon_{ijk} x_j y_k = (x \times y)_j.$$

Hence, $\Lambda : (x \otimes y)$ stands for the vector product of $x$ and $y$.

In general, the following proposition is easy to verify.

**Theorem 2.2.10** Within the set of all tensors of equal order $q \in \mathbb{N}$, the double-dot product satisfies the properties of an inner product, as it will be defined in Definition 2.5.1.

Note that the usual multiplication of matrices $S$ and $T$ fulfills $ST = (\sum_{j=1}^3 S_{ij} T_{jk})_{i, k=1, 2, 3} = \text{tr}_{23} (S \otimes T)$.

We need another notation in the context of tensors.